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# Some Results on Analysis, Synthesis and Frame Operators Associated with a Fixed Element in 2-Hilbert Spaces

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**Abstract:** This paper develops the operator-theoretic framework of $\xi$-frames in 2-Hilbert spaces through the analysis, synthesis, and $\xi$-frame operators associated with a fixed element. The $\xi$-analysis operator is introduced for $\xi$-Bessel sequences, and its linearity and boundedness are established. Since $\xi$-frame coefficients depend only on equivalence classes determined by the fixed element, the construction is formulated on the corresponding quotient Hilbert space. The $\xi$-synthesis operator is then obtained as the adjoint of the quotient analysis operator and lifted to the original 2-Hilbert space by means of a complementary subspace. Furthermore, the $\xi$-frame operator is defined as the composition of the synthesis and analysis operators, and its boundedness, self-adjointness, positivity, and operator form of the $\xi$-frame inequalities are proved. The invertibility of the induced $\xi$-frame operator leads naturally to the canonical dual $\xi$-frame and the associated reconstruction formula. A finite-dimensional example is presented to illustrate the theoretical results. These findings provide a rigorous foundation for reconstruction theory and further investigations of generalized frame structures in 2-Hilbert spaces.

**Keywords:** 2-Hilbert space; $\xi$-frame; $\xi$-Bessel sequence; analysis operator; synthesis operator; frame operator; canonical dual frame. **MSC 2020:** 46C05, 42C15, 47B99.

## 1. INTRODUCTION

The theory of frames [2], introduced by Duffin and Schaeffer and later developed extensively by Christensen [1], provides a redundant yet stable representation of vectors in Hilbert spaces. Besides their theoretical importance, frames have become indispensable in signal processing, harmonic analysis, sampling theory, and data reconstruction. The theory of 2-inner product spaces originated with the pioneering work of Gähler [3,4], who introduced it as a natural generalization of the classical inner product framework. This approach subsequently led to the development of 2-normed and 2-Hilbert spaces, extending many geometric and functional analytic aspects of Hilbert space theory. Within this framework, a fixed nonzero element $\xi$ induces the inner product $\langle (x,\xi),(y,\xi)\rangle$ and the corresponding seminorm $\| x,\xi \|$. This fixed-element formulation provides the foundation for extending classical frame theory to 2-Hilbert spaces, allowing the analysis, synthesis, and frame operators to be developed in an analogous operator-theoretic setting. Among the most fundamental operators in frame theory are the analysis and synthesis operators. These operators transform vectors into coefficient sequences and reconstruct vectors from those sequences, respectively. Their interaction leads to the frame operator, which plays a central role in reconstruction formulas and dual frame theory. The objective of this paper is to develop the operator theory of $\xi$-frames in 2-Hilbert spaces. We establish the boundedness of the $\xi$-analysis and $\xi$-synthesis operators, identify their adjoint relationship, and derive the associated $\xi$-frame operator.

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## II. PRELIMINARIES

Throughout this paper, $H$ denotes a 2-Hilbert space and $\xi$ is a fixed nonzero element of $H$. All inner products and norms are taken with respect to the fixed element $\xi$.

**Definition 2.1[3]:** Let $H$ be a complex vector space. A complex-valued mapping $\langle(\cdot,\cdot),(\cdot,\cdot)\rangle: H^2 \times H^2 \to \mathbb{C}$ is called a 2-inner product if, for every $a,b,c,d,e \in H$ and $\alpha,\lambda \in \mathbb{C}$, the following conditions are satisfied.
(i) If $a$ and $b$ are linearly independent, then $\langle(a,b),(a,b)\rangle > 0$.
(ii) $\langle(a,b),(c,d)\rangle = \overline{\langle(c,d),(a,b)\rangle}$
(iii) $\langle(a,b),(c,d)\rangle = -\overline{\langle(b,a),(c,d)\rangle}$
(iv) $\langle(\alpha a,b),(c,d)\rangle = \alpha\langle(a,b),(c,d)\rangle$.
(v) $\langle(a+e,b),(c,d)\rangle = \langle(a,b),(c,d)\rangle + \langle(e,b),(c,d)\rangle$.
The pair $(H,\langle(\cdot,\cdot),(\cdot,\cdot)\rangle)$ is called a 2-inner product space.

**Remark 2.2 ([3]):** The axioms of Definition 2.1 immediately imply the following identities. For every $a,b,c,d,\bar{b},\bar{c},\bar{d} \in H$ and $\alpha,\beta \in \mathbb{C}$,

$$\begin{array}{ll}
(i) & \langle(a,\alpha b+\bar{b}),(c,d)\rangle = \alpha\langle(a,b),(c,d)\rangle + \langle(a,\bar{b}),(c,d)\rangle, \\
(ii) & \langle(a,b),(\beta c+\bar{c},d)\rangle = \bar{\beta}\,\langle(a,b),(c,d)\rangle + \langle(a,b),(\bar{c},d)\rangle, \\
(iii) & \langle(a,b),(c,\beta d+\bar{d})\rangle = \bar{\beta}\,\langle(a,b),(c,d)\rangle + \langle(a,b),(c,\bar{d})\rangle, \\
(iv) & \langle(a,b),(c,d)\rangle = \langle(b,a),(d,c)\rangle, \\
(v) & \langle(a,b),(a,b)\rangle = 0 \iff a,b \text{ are linearly dependent}, \\
(vi) & \langle(a,b),(a,b)\rangle \geq 0.
\end{array}$$

**Theorem 2.3 ([3]):** Let $H$ be a 2-inner product space. Then the mapping $\|\cdot,\cdot\|: H \times H \to \mathbb{R}$ defined by $\| a,b \| = \sqrt{\langle(a,b),(a,b)\rangle}$ is a 2-norm on $H$.

**Definition 2.4 ([3,8]):** Let $0 \neq \xi \in H$. A sequence $\{x_n\} \subset H$ is called a $\xi$-Cauchy sequence if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\| x_m - x_n, \xi \| < \varepsilon, \forall m,n \geq N$.
If every $\xi$-Cauchy sequence converges in the semi norm $\|\cdot,\xi\|$, then $H$ is called a $\xi$-Hilbert space.
If $H$ is a $\xi$-Hilbert space for every nonzero vector $\xi \in H$, then $H$ is called a 2-Hilbert space.

Throughout the remainder of this paper, $\xi$ denotes a fixed nonzero element of the 2-Hilbert space $H$. Accordingly, the 2-inner product $\langle(x,\xi),(y,\xi)\rangle$ and the corresponding 2-norm $\| x,\xi \| = \sqrt{\langle(x,\xi),(x,\xi)\rangle}$ will be used throughout without further mention. Observe that $\| x,\xi \| = 0 \iff x \in \text{span}\{\xi\}$, and therefore $\|\cdot,\xi\|$ is a seminorm on $H$.

**Proposition 2.5:** Define a relation $\sim$ on $H$ by $x \sim y \iff x - y \in \text{span}\{\xi\}$. Then $\sim$ is an equivalence relation on $H$. Consequently, the quotient space $H_\xi := H/\text{span}\{\xi\}$ is a vector space. For each $x \in H$, we denote by $[x] = x + \text{span}\{\xi\} = \{x + \lambda\xi : \lambda \in \mathbb{F}\}$ the equivalence class of $x$.

**Proposition 2.6:** The mapping $\langle[x],[y]\rangle = \langle x + \text{span}\{\xi\}, y + \text{span}\{\xi\}\rangle = \langle(x,\xi),(y,\xi)\rangle$, is well defined and defines an inner product on the quotient space $H_\xi = \frac{H}{\text{span}\{\xi\}}$.

**Remark 2.7:** The quotient space $H/\text{span}\{\xi\}$, equipped with the above inner product, is a Hilbert space.

**Remark 2.8.** For every $u \in H$ and every scalar $\lambda$, $\| u + \lambda\xi, \xi \| = \| u,\xi \|$.

**Definition 2.9. [6,7,9] $\xi$-Bessel Sequence** A sequence $\{x_j\}_{j\in J} \subset H$ is called a $\xi$-Bessel sequence if there exists a constant $B > 0$ such that

$$\sum_{j\in J} |\langle(x,\xi),(x_j,\xi)\rangle|^2 \leq B \| x,\xi \|^2, \ \forall x \in H.$$

The constant B is called a $\xi$-Bessel bound.

**Definition 2.10 [6,7,9] $\xi$-Frame**

A sequence $\{x_j\}_{j\in J} \subset H$ is said to be a $\xi$-frame for H if there exist constants $0 < A \leq B < \infty$ satisfying

$$A \| x,\xi \|^2 \leq \sum_{j\in J} |\langle(x,\xi),(x_j,\xi)\rangle|^2 \leq B \| x,\xi \|^2, \ \forall x \in H.$$

The positive numbers A and B are called the lower and upper $\xi$-frame bounds, respectively.

**Theorem 2.11[8]: Riesz Representation Theorem**

Let H be a Hilbert space and let $f: H \to \mathbb{R}$ be a bounded linear functional. Then there exists a unique vector $y \in H$ such that

$$f(x) = \langle x,y\rangle, \ \forall x \in H.$$

Moreover,

$$\| f \| = \| y \|.$$

## III. ANALYSIS OPERATOR ASSOCIATED WITH A $\xi$-BESSEL SEQUENCE

The analysis operator converts each vector of the 2-Hilbert space into its sequence of $\xi$-frame coefficients. Since vectors differing by an element of $span\{\xi\}$ possess identical coefficients, the construction naturally extends to the quotient Hilbert space.

**Definition 3.1. Analysis Operator** Let $\{x_j\}_{j\in J}$ be a $\xi$-Bessel sequence in H. The analysis operator $T_\xi: H \longrightarrow \ell^2(J)$ is defined by $T_\xi x = \{\langle(x,\xi),(x_j,\xi)\rangle\}_{j\in J}, x \in H$. Each vector is therefore represented by its $\xi$-frame coefficient sequence.

**Remark 3.2:** Since $\{x_j\}_{j\in J}$ satisfies the $\xi$-Bessel inequality, the sequence $T_\xi(x)$ belongs to $\ell^2(J)$ for every $x \in H$. Consequently, the analysis operator is well defined.

**Theorem 3.3. Linearity** The analysis operator $T_\xi$ is linear.

**Proof:** Let $x, y \in H$ and $\alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned} T_\xi(\alpha x + \beta y) &= \{\alpha\langle(x,\xi),(x_j,\xi)\rangle + \beta\langle(y,\xi),(x_j,\xi)\rangle\}_{j\in J} \\ &= \alpha\{\langle(x,\xi),(x_j,\xi)\rangle\}_{j\in J} + \beta\{\langle(y,\xi),(x_j,\xi)\rangle\}_{j\in J} \\ &= \alpha T_\xi x + \beta T_\xi y. \end{aligned}$$

Hence, $T_\xi$ is linear.

**Theorem 3.4. Boundedness**

Let $\{x_j\}_{j\in J}$ be a $\xi$-Bessel sequence with upper Bessel bound $B$. Then

$$\| T_\xi \| \leq \sqrt{B}.$$

**Proof:** For every $x \in H$,

$$\begin{aligned} \|T_\xi x\|_{\ell^2}^2 &= \sum_{j\in J} |\langle(x,\xi),(x_j,\xi)\rangle|^2 \\ &\leq B\|x,\xi\|^2 \text{using (2.9)} \end{aligned}$$

by the $\xi$-Bessel inequality. Therefore,

$$\| T_\xi(x) \| \leq \sqrt{B} \ \| x,\xi \|.$$

Taking the supremum over all nonzero vectors' yields

$$\| T_\xi \| \leq \sqrt{B}.$$

Thus, the analysis operator is bounded.

**Remark 3.5. Motivation for the Quotient Construction**

The operator $T_\xi$ is generally not injective. Indeed, if $y = x + \lambda\xi$, $\lambda \in \mathbb{R}$, then $\langle(y,\xi),(x_j,\xi)\rangle = \langle(x,\xi),(x_j,\xi)\rangle, \forall j \in J$ because every multiple of $\xi$ has zero $\xi$-seminorm. Hence, $T_\xi(y) = T_\xi(x)$. Therefore, the analysis operator depends only on the equivalence class of a vector modulo span $\{\xi\}$, which motivates its formulation on the quotient Hilbert space.

## IV. SYNTHESIS OPERATOR ASSOCIATED WITH A $\xi$-FRAME

The synthesis operator reconstructs vectors from square-summable coefficient sequences. Since the analysis operator depends only on equivalence classes modulo $span\{\xi\}$, its adjoint is first defined on the quotient Hilbert space and then identified with the original 2-Hilbert space through a fixed complementary subspace.

**Definition 4.1. Quotient Synthesis Operator**

Let $\tilde{T}_\xi: \frac{H}{\text{span}\{\xi\}} \longrightarrow \ell^2(J)$ be the quotient analysis operator. The quotient synthesis operator is defined as its Hilbert-space adjoint, $\tilde{T}_\xi^*: \ell^2(J) \longrightarrow H/\text{span}\{\xi\}$, satisfying $\langle\tilde{T}_\xi([x]), c\rangle_{\ell^2} = \langle[x], \tilde{T}_\xi^* c\rangle_{H_\xi}$ for every equivalence class $[x]$ and every coefficient sequence $\{c_j\} \in \ell^2(J)$.

**Theorem 4.2. Existence and Uniqueness**

There exists a unique bounded quotient synthesis operator $\tilde{T}_\xi^*: \ell^2(J) \to H/\text{span}\{\xi\}$.

**Proof:** The quotient space $H/\text{span}\{\xi\}$ and $\ell^2(J)$ are Hilbert spaces, and the quotient analysis operator is bounded. Therefore, the Riesz Representation Theorem guarantees the existence of a unique bounded adjoint operator $\tilde{T}_\xi^*$.

**Theorem 4.3. Representative Family of a $\xi$-Frame**

Let M be a fixed complementary subspace of span$\{\xi\}$ such that $H = M \oplus \text{span}\{\xi\}$.

If $\{x_j\}_{j\in J}$ is a $\xi$-frame for H, then there exists a unique family $\{\hat{x}_j\}_{j\in J} \subset M$ satisfying $[\hat{x}_j] = [x_j]$, $j \in J$. Moreover, $\{\hat{x}_j\}$ is a $\xi$-frame with the same frame bounds as $\{x_j\}$.

**Proof:** Since $H = M \oplus \text{span}\ \{\xi\}$, each frame element admits a unique decomposition

$$x_j = \hat{x}_j + \lambda_j\xi,$$

where $\hat{x}_j \in M$. Consequently,

$$[x_j] = [\hat{x}_j].$$

Furthermore,

$$\langle(x,\xi),(x_j,\xi)\rangle = \langle(x,\xi),(\hat{x}_j,\xi)\rangle$$

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for every $x \in H$. Hence the $\xi$-frame coefficients remain unchanged, and the ξ-frame inequalities are preserved. Therefore, $\{\hat{x}_j\}$is also a $\xi$-frame with identical frame bounds. The uniqueness follows from the direct sum decomposition.

**Definition 4.4. Synthesis Operator**

After identifying each equivalence class with its unique representative in M, the synthesis operator is defined by

$$T_\xi^*: \ell^2(J) \longrightarrow H$$

with

$$T_\xi^*(\{c_j\}) = \sum_{j \in J} c_j \hat{x}_j.$$

This operator provides the lifting of the quotient synthesis operator to the original 2-Hilbert space.

**Theorem 4.5. Linearity**

The synthesis operator $T_\xi^*$is linear.

**Proof:** Let $\{a_j\}, \{b_j\} \in \ell^2(J)$and $\alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned} T_\xi^*(\alpha\{a_j\} + \beta\{b_j\}) &= \sum_{j \in J} (\alpha a_j + \beta b_j)\hat{x}_j \\ &= \alpha \sum_{j \in J} a_j \hat{x}_j + \beta \sum_{j \in J} b_j \hat{x}_j \\ &= \alpha T_\xi^*(\{a_j\}) + \beta T_\xi^*(\{b_j\}). \end{aligned}$$

Hence, $T_\xi^*$is linear.

**Theorem 4.6. Boundedness of the Synthesis Operator**

The synthesis operator is bounded and satisfies

$$\| T_\xi^* \| \leq \sqrt{B},$$

where B is the upper $\xi$-frame bound.

**Proof:** The operator $T_\xi^*$is obtained by lifting the bounded quotient synthesis operator. Since adjoint operators on Hilbert spaces possess identical norms,

$$\| T_\xi^* \| = \| \tilde{T}_\xi^* \| = \| \tilde{T}_\xi \| \leq \sqrt{B}.$$

Therefore, the synthesis operator is bounded.

**Theorem 4.7. $\xi$-Adjoint Relation**

The synthesis operator is the $\xi$-adjoint of the analysis operator; namely,

$$\langle T_\xi x, \{c_j\} \rangle_{l^2} = \langle (x, \xi), (T_\xi^* \{c_j\}, \xi) \rangle$$

for every $x \in H$ and $\{c_j\} \in \ell^2(J)$.

**Proof:** Using the definition of the synthesis operator,

Let $c = \{c_j\}$, then

$$\begin{aligned} \langle T_\xi x, c \rangle_{l^2} &= \sum_{j \in J} \langle (x, \xi), (x_j, \xi) \rangle \bar{c}_j \\ &= \langle (x, \xi), (\sum_{j \in J} c_j x_j, \xi) \rangle \\ &= \langle (x, \xi), (T_\xi^* c, \xi) \rangle \end{aligned}$$

Hence, $T_\xi^*$is the $\xi$-adjoint of $T_\xi$.

## V. $\xi$-FRAME OPERATOR

The frame operator is obtained by composing the synthesis and analysis operators. It provides the operator-theoretic representation of a $\xi$-frame and forms the basis for reconstruction in the associated quotient Hilbert space.

**Definition 5.1. $\xi$-Frame Operator**

Let $\{x_j\}_{j \in J}$be a ξ-frame with analysis operator $T_\xi$and synthesis operator $T_\xi^*$. The $\xi$-frame operator is defined by $S_\xi = T_\xi^* T_\xi$. For every $x \in H$,

$$S_\xi x = \sum_{j \in J} \langle (x, \xi), (x_j, \xi) \rangle x_j.$$

This operator maps each vector to its reconstruction from the $\xi$-frame coefficients.

**Theorem 5.2. Boundedness of the $\xi$-Frame Operator**

The $\xi$-frame operator is bounded and satisfies

$$\| S_\xi \| \leq B,$$

where B denotes the upper ξ-frame bound.

**Proof:** Since both $T_\xi$and $T_\xi^*$are bounded,

$$\begin{aligned} \| S_\xi \| &= \| T_\xi^* T_\xi \| \\ &\leq \| T_\xi^* \| \ \| T_\xi \| \\ &\leq \sqrt{B} \cdot \sqrt{B} = B. \end{aligned}$$

Hence, $S_\xi$ is a bounded linear operator.

**Theorem 5.3. Self-Adjointness**

The $\xi$-frame operator is self-adjoint; that is, $\langle (S_\xi x, \xi), (y, \xi) \rangle = \langle (x, \xi), (S_\xi y, \xi) \rangle$ for every $x, y \in H$.

**Proof:** Using the $\xi$-adjoint relation established in Theorem 4.7,

$$\begin{aligned} \langle (S_\xi x, \xi), (y, \xi) \rangle &= \langle (T_\xi^*(T_\xi x), \xi), (y, \xi) \rangle \\ &= \overline{\langle (y, \xi), (T_\xi^*(T_\xi x), \xi) \rangle} \\ &= \overline{\langle T_\xi y, T_\xi x \rangle}_{l^2} \\ &= \langle T_\xi x, T_\xi y \rangle_{l^2} \end{aligned}$$

Similarly,

$$\langle (x, \xi), (S_\xi y, \xi) \rangle = \langle (x, \xi), (T_\xi^*(T_\xi y), \xi) \rangle = \langle T_\xi x, T_\xi y \rangle_{l^2}$$

Therefore,

$\langle (S_\xi x, \xi), (y, \xi) \rangle = \langle (x, \xi), (S_\xi y, \xi) \rangle$ Consequently, $S_\xi$ is self-adjoint.

**Theorem 5.4. Positivity**

The $\xi$-frame operator is positive; namely, $\langle (S_\xi x, \xi), (y, \xi) \rangle \geq 0$, $\forall x \in H$.

**Proof:** For an arbitrary vector $x \in H$, $\langle (S_\xi x, \xi), (y, \xi) \rangle = \langle (T_\xi^*(T_\xi x), \xi), (y, \xi) \rangle = \langle T_\xi x, T_\xi y \rangle = \| T_\xi x \|^2$

Since the squared norm in $\ell^2(J)$ is always non-negative, the result follows.

**Theorem 5.5. Operator Form of the $\xi$-Frame Inequalities**

Let A and B be the $\xi$-frame bounds of $\{x_j\}_{j \in J}$. Then

$$A\, I_\xi \leq S_\xi \leq B\, I_\xi,$$

with respect to the operator order induced by the fixed element $\xi$.

**Proof:** The $\xi$-frame inequality gives

$$A \| x, \xi \|^2 \leq \sum_{j \in J} |\langle (x, \xi), (x_j, \xi) \rangle|^2 \leq B \| x, \xi \|^2.$$

Since

$$\sum_{j \in J} |\langle (x, \xi), (x_j, \xi) \rangle|^2 = \langle (S_\xi x, \xi), (x, \xi) \rangle$$

we obtain

$$A \| x, \xi \|^2 \leq \langle (S_\xi x, \xi), (x, \xi) \rangle \leq B \| x, \xi \|^2.$$

As this holds for every vector in H, it is equivalent to the operator inequality

$$A\, I_\xi \leq S_\xi \leq B\, I_\xi.$$

Hence, the $\xi$-frame inequalities admit the above operator formulation.

**Remark 5.6:** The inequality $A\, I_\xi \leq S_\xi \leq B\, I_\xi$ is the operator-theoretic counterpart of the $\xi$-frame condition. In particular, the lower bound shows that the induced operator on the quotient Hilbert space is bounded below, while the upper bound guarantees continuity. These properties are fundamental for the reconstruction theory developed from the $\xi$-frame operator.

## VI. RECONSTRUCTION THEOREM AND CANONICAL DUAL $\xi$-FRAME

The operator properties established in the previous section imply that the $\xi$-frame operator is invertible on the quotient Hilbert space. This leads naturally to the canonical dual $\xi$-frame and the corresponding reconstruction formula. The results below are adapted from Section 4.2 of the thesis.

**Theorem 6.1. Invertibility of the $\xi$-Frame Operator**

Let $S_\xi$ be the $\xi$-frame operator associated with a $\xi$-frame whose frame bounds are $A$ and $B$. Then the induced operator on the quotient space,

$$\tilde{S}_\xi : H/\text{span}\{\xi\} \longrightarrow H/\text{span}\{\xi\},$$

is bounded, positive, and invertible. Moreover,

$$B^{-1} I \leq \tilde{S}_\xi^{-1} \leq A^{-1} I.$$

**Proof:** From the operator form of the $\xi$-frame inequalities,

$$AI \leq \tilde{S}_\xi \leq BI.$$

Hence, $\tilde{S}_\xi$ is bounded below by a positive operator and is therefore injective with closed range. Since the $\xi$-frame inequalities hold on the entire quotient Hilbert space, its range is all of $H/\text{span}\{\xi\}$. Consequently, $\tilde{S}_\xi$ is boundedly invertible. Applying inversion to the operator inequality yields $B^{-1} I \leq \tilde{S}_\xi^{-1} \leq A^{-1} I$. Thus, the induced $\xi$-frame operator is positive and invertible.

**Definition 6.2. Canonical Dual $\xi$-Frame**

Let $\{x_j\}_{j \in J}$ be a $\xi$-frame with induced frame operator $\tilde{S}_\xi$. The family

$$\{\tilde{S}_\xi^{-1}[x_j]\}_{j \in J}$$

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is called the canonical dual $\xi$-frame. After selecting representatives from a fixed complementary subspace, it may be written as $\tilde{x}_j = S_\xi^{-1} x_j,\ j \in J$.The sequence $\{\tilde{x}_j\}$is referred to as the canonical dual of the $\xi$-frame.

**Theorem 6.3. Reconstruction Formula**

Let $\{x_j\}_{j\in J}$be a $\xi$-frame for H, and let $\{\tilde{x}_j\}_{j\in J}$denote its canonical dual $\xi$-frame. Then every vector $x \in H$ admits the reconstruction

$$x = \sum_{j\in J} \langle (x,\xi), (\tilde{x}_j,\xi) \rangle\, x_j$$

Equivalently,

$$x = \sum_{j\in J} \langle (x,\xi), (x_j,\xi) \rangle\, \tilde{x}_j$$

**Proof:** Since $\tilde{S}_\xi$is invertible,

$$x = S_\xi^{-1} S_\xi x.$$

Using the representation of the $\xi$-frame operator,

$$S_\xi x = \sum_{j\in J} \langle (x,\xi), (x_j,\xi) \rangle\, x_j$$

we obtain

$$\begin{aligned} x &= S_\xi^{-1}\left( \sum_{j\in J} \langle (x,\xi), (x_j,\xi) \rangle\, x_j \right) \\ &= \sum_{j\in J} \langle (x,\xi), (x_j,\xi) \rangle S_\xi^{-1}\, x_j \\ &= \sum_{j\in J} \langle (x,\xi), (x_j,\xi) \rangle\, \tilde{x}_j \end{aligned}$$

Since $S_\xi$is self-adjoint, the symmetric expansion follows immediately:

$$x = \sum_{j\in J} \langle (x,\xi), (\tilde{x}_j,\xi) \rangle\, x_j$$

Therefore, every vector possesses a stable reconstruction with respect to both the original $\xi$-frame and its canonical dual.

**Theorem 6.4. Canonical Dual is a ξ-Frame**

The canonical dual sequence

$$\{\tilde{x}_j\}_{j\in J} = \{S_\xi^{-1} x_j\}_{j\in J}$$

forms a $\xi$-frame for H.

**Proof:** For each $x \in H$, let

$$y = S_\xi^{-1} x.$$

By the self-adjointness of $S_\xi^{-1}$,

$$\langle (x,\xi), (\tilde{x}_j,\xi) \rangle = \langle (y,\xi), (x_j,\xi) \rangle$$

Applying the ξ-frame inequalities to the vector $y$ gives

$$A \parallel y,\xi \parallel^2 \le \sum_{j\in J} |\langle (y,\xi), (x_j,\xi) \rangle|^2 \le B \parallel y,\xi \parallel^2.$$

Substituting the above identity yields

$$A \parallel S_\xi^{-1} x,\xi \parallel^2 \le \sum_{j\in J} |\langle (x,\xi), (\tilde{x}_j,\xi) \rangle|^2 \le B \parallel S_\xi^{-1} x,\xi \parallel^2.$$

Hence, $\{\tilde{x}_j\}_{j\in J}$satisfies the $\xi$-frame inequalities and is therefore a $\xi$-frame. This section completes the reconstruction theory by establishing the invertibility of the $\xi$-frame operator, introducing the canonical dual $\xi$-frame, and deriving the corresponding reconstruction identities.

## VII. ILLUSTRATIVE EXAMPLE

This example demonstrates the construction of the analysis operator, synthesis operator, and $\xi$-frame operator for a finite-dimensional 2-Hilbert space.

**Example 7.1**

Let

$$H = \mathbb{R}^3,\ \xi = (0,0,1).$$

The fixed element $\xi$ induces the inner product

$$\langle (x,\xi), (y,\xi) \rangle = x_1 y_1 + x_2 y_2$$

and the corresponding norm

$$\parallel x,\xi \parallel = \sqrt{x_1^2 + x_2^2}.$$

Choose the family

$$x_1 = (1,0,0),\ x_2 = (0,1,0)$$

Since these vectors form an orthonormal basis of the quotient space $H/\text{span}\,\{\xi\}$, they constitute a $\xi$-frame with frame bounds

$$A = B = 1.$$

**Analysis operator**

For an arbitrary vector

$$x = (a,b,c),$$

the $\xi$-frame coefficients are

$$\langle (x,\xi),(x_1,\xi)\rangle = a, \langle (x,\xi),(x_2,\xi)\rangle = b.$$

Hence,

$$T_\xi(a,b,c) = (a,b)$$

The analysis operator extracts precisely the non-degenerate coordinates determined by the fixed element $\xi$.

**Synthesis operator**

For a coefficient sequence $(\alpha,\beta) \in \ell^2(\{1,2\})$,

$$T_\xi^*(\alpha,\beta) = \alpha x_1 + \beta x_2$$
$$= (\alpha,\beta,0).$$

Thus, the synthesis operator reconstructs the representative lying in the complementary subspace

$$M = \{(u,v,0): u,v \in \mathbb{R}\}.$$

**$\xi$-frame operator**

The frame operator is

$$S_\xi = T_\xi^* T_\xi.$$

Therefore,

$$S_\xi(a,b,c) = T_\xi^*(a,b) = (a,b,0)$$

Its matrix representation is

$$S_\xi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

**Verification of operator properties**

For every $x = (a,b,c) \in H$,

$$\langle (S_\xi x,\xi),(y,\xi)\rangle = a^2 + b^2 = \| x,\xi \|^2 \geq 0$$

Hence, $S_\xi$ is positive. Moreover,

$$\langle (S_\xi x,\xi),(y,\xi)\rangle = \langle (x,\xi),(S_\xi y,\xi)\rangle$$

showing that it is self-adjoint. Since $A = B = 1$, the operator satisfies

$$I_\xi \leq S_\xi \leq I_\xi,$$

and therefore $S_\xi$ acts as the identity on the quotient Hilbert space. This example illustrates that the analysis operator extracts $\xi$-frame coefficients, the synthesis operator reconstructs vectors from those coefficients, and the $\xi$-frame operator realizes the orthogonal projection onto the effective quotient space.

## VIII. CONCLUSION

This paper developed the operator-theoretic framework of $\xi$-frames in 2-Hilbert spaces through the analysis, synthesis, and $\xi$-frame operators. The $\xi$-analysis operator was introduced as a bounded linear mapping into the coefficient space, while the $\xi$-synthesis operator was obtained as its adjoint through the quotient Hilbert space construction. Their composition produced the $\xi$-frame operator, whose boundedness, self-adjointness, positivity, and operator inequalities were established. The invertibility of the induced operator naturally led to the canonical dual $\xi$-frame and the associated reconstruction formula. The finite-dimensional example demonstrates the practical realization of these operators and confirms the theoretical results. Together, these constructions provide a foundation for further investigations of generalized frame operators and operator-controlled frame theory in 2-Hilbert spaces.

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**Author contribution Statement**

Conceptualization: G.Upender Reddy; Literature Review and Methodology design: B.S.S.P. Rajasekhar ; Formal analysis: B.S.S.P. Rajasekhar ; Investigation: B.S.S.P. Rajasekhar ; Resources: G.Upender Reddy and B.S.S.P. Rajasekhar ; Data Curation: B.S.S.P. Rajasekhar ; Writing original draft preparation: B.S.S.P. Rajasekhar; Writing review and editing: G.Upender Reddy; Visualization: B.S.S.P. Rajasekhar ;Supervision: G.Upender Reddy; Project Administration: G.Upender Reddy. All authors have read and agreed to the published version of the manuscript

**Conflict of interest**

The authors declare no conflicts of interest.

**REFERENCES**


1. Christensen, O. (2016). An Introduction to Frames and Riesz Bases (2nd ed.). Birkhäuser, Basel. https://doi.org/10.1007/978-3-319-25613-9
2. Duffin, R. J., & Schaeffer, A. C. (1952). A class of nonharmonic Fourier series. Transactions of the American Mathematical Society, 72(2), 341–366. https://doi.org/10.1090/S0002-9947-1952-0047179-6
3. Eshaghi Gordji, M., Divandari, A., Safi, M. R., & Cho, Y. J. (2013). On reformations of 2-Hilbert spaces. The Australian Journal of Mathematical Analysis and Applications, 10(1), Article 8, 1–11.
4. Gähler, S. (1964). Lineare 2-normierte Räume. Mathematische Nachrichten, 28, 1–43. https://doi.org/10.1002/mana.19640280102
5. Gähler, S. (1965). Über 2-Banach-Räume. Mathematische Nachrichten, 42, 335–347. https://doi.org/10.1002/mana.19690420414
6. Rajasekhar, B. S. S. P., & Upender Reddy, G. (2026). Some Results on Frame Operators in 2-Hilbert Spaces. Boletim da SociedadeParanaense de Matemática, 44(17), 1–15. https://doi.org/10.5269/bspm.82362
7. Rajasekhar, B. S. S. P., & Upender Reddy, G. (2026). Some Results on Generalized Frame Operators Associated with a Fixed Element in 2-Hilbert Spaces. IAENG International Journal of Applied Mathematics, 56(4), 1446–1453.
8. Rudin, W. (1991). Functional Analysis (2nd ed.). McGraw–Hill, New York.
9. Upender Reddy, G. (2025). On the Properties of Frames in 2-Hilbert Spaces. Asian Research Journal of Mathematics, 21(4), 136–146. Article No. ARJOM.133942. https://doi.org/10.9734/arjom/2025/v21i4916
10. Upender,Naresh,Narsimha(2026), Certain Applications via $\varphi$-Contraction Type Coupled Fixed Points in $C^*$-Algebra Valued $S_b$-Metric Spaces ,IRJCS:International Research Journal of Computer Science,Volume 13, Issue 02 of 2026 pages 77-85 https://doi.org/10.26562/irjcs.2026.v1302.02
11. Upender,Sravani,Mounika,Kavya,Rajeshwari(2026),Solutions of Interval Linear Systems Using Interval Gauss–Jacobi and Interval Gauss–Seidel Iterative Methods, IJIRAE: International Journal of Innovative Research in Advanced Engineering, Volume 13, Issue 06 of 2026 pages 882-889 https://doi.org/10.26562/ijirae.2026.v1307.01
12. Upender,Varsha,Chamanthi,Shivaranjani(2026),A Study of the Interval Newton Method for Nonlinear Equations. IJIRIS: International Journal of Innovative Research in Information Security, Volume 12, Issue 05 of 2026 pages 327-331 https://doi.org/10.26562/ijiris.2026.v1205.01