IRJCS: International Research Journal of Computer Science
Volume 13, Issue 02, February 2026
ISSN: 2393-9842
https://www.irjcs.com/archives
https://doi.org/10.26562/irjcs.2025.v1302.02

Certain Applications via φ-Contraction Type Coupled Fixed Points in C^*-Algebra Valued S_b-Metric Spaces

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Publication History
Manuscript Reference: IRJCS/RS/Vol.13/Issue02/CSFB26.FBCS10081
Research Article | Open Access | Double-Blind Peer Reviewed Article ID: IRJCS/RS/Vol.13/Issue02/CSFB26.FBCS10081
Received: 13, December 2025, Revised: 10, January 2026, Accepted: 03 February 2026 Published Online: 02 March 2026
https://www.irjcs.com/volumes/Vol13/iss-02/02.CSFB26.FBCS10081.pdf

Article Citation: Upender,Naresh,Narsimha(2026), Certain Applications via φ-Contraction Type Coupled Fixed Points in C*-Algebra Valued S_b-Metric Spaces , IRJCS:International Research Journal of Computer Science, Volume 13, Issue 02 of 2026 pages 77-85 **Doi:>** https://doi.org/10.26562/irjcs.2026.v1302.02

BibTeX Key Upender@2026Certain

IRJCS papers should be cited as IRJCS (International Research Journal of Computer Science, AM Publications, India 2026, ISSN 2393-9842, https://doi.org/10.26562/ijrcs.2025.v1302.02 The journal's official abbreviation is IRJCS.

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Abstract: In this paper, we establish unique common coupled fixed-point findings for two pairs of ω-compatible mappings in C*-algebra-valued S_b- metric spaces that meet φ-generalized contractive conditions. Also, we provide an example to back up the findings we were able to get. Additionally, the study offers an application to demonstrate homotopy and the existence and uniqueness of a solution for a non-linear integral problem.

Keywords: $C^*-algebra$ Valued Sb-metric Spaces, ω-compatible mappings, φ-generalized weakly contractive mappings, non-linear integral equations, homotopy.
2000 Mathematics Subject Classification: 2010 MSC :54H25,47H10,54E50.

INTRODUCTION

Polish mathematician Stephen Banach developed the most notable fixed- point theorem in 1992. It is often referred to as the Banach contraction principle and is used extensively in a variety of mathematical contexts. The study of fixed- point theory in metric spaces is thought to have led to the discovery of the Banach contraction principle. Different generalizations of metric spaces were introduced by a number of mathematicians worldwide. The concept of C^*-algebra-valued metric spaces was first introduced by Ma et al. in 2014. They presented the notion of C^*-algebra-valued b-metric spaces in 2015 and looked at a few results. In order to find similar principles, Razavi and Masiha also explored C^*-algebra-valued b-metric spaces [1].Combining the ideas of S and b-metric spaces, Sedghi et al. [2] established S_b-metric spaces and proved common fixed-point results in these spaces. Several authors have produced a wealth of results on S_b-metric spaces (e.g.,[3],[4]) principles) in an effort to improve. Razavi et al. introduced the concept of C^*-algebra-valued S_b-metric space [11] in 2023, drawing inspiration from the work of Souayah and Mlaiki in [3]. They also established some common fixed point results in this space [12]. Guo and Lakshmikantham [13] introduced the concept of coupled fixed point for the first time in 1987. Later, Bhaskar and Lakshmikantham constructed a novel fixed point theorem for a mixed monotone mapping in a metric space driven with partial ordering by using a weak contractivity type assumption. The concept of weak compatibility was first presented by Jungck and Rhoades in 1998. They showed that while compatible mappings are weakly compatible, the opposite is not true. For more study findings on coupled fixed point outcomes, see ([14]-[15]) and associated references.

This work presents coupled fixed point results for two pairs of ω-compatible mappings that meet φ-generalized contractive requirements in C^*-algebra-valued S_b-metric spaces. We can also provide appropriate and relevant examples for both integral equations and homotopy. First, we recall some basic results.

1. PRELIMINARIES

This section provides a short introduction to some realities about the theory of C^*-algebras. First, suppose that $\mathfrak{U}$ is a unital C^*-algebra with the unit $1_{\mathfrak{U}}$ .Set $\mathfrak{U}_h$={s∈ $\mathfrak{U}$; s=s^* } .The element s∈ $\mathfrak{U}$ is said to be positive ,and we write s≥ $0_{\mathfrak{U}}$ if and only if s=s^* and $\sigma(s) \subseteq [0,\infty)$, *in which* $0_{\mathfrak{U}}$ in $\mathfrak{U}$ is the zero element and spectrum of s is $\sigma(s)$ .On $\mathfrak{U}_h$,we can find a natural partial ordering given by $\ell \preccurlyeq \varphi$ if and only if $\varphi - \ell \succcurlyeq 0_{\mathfrak{U}}$ .we denote with $\mathfrak{U}_+$= {s∈ $\mathfrak{U}$; s≥ $0_{\mathfrak{U}}$} and $\mathfrak{U}'$={s∈ $\mathfrak{U}$; st=ts $\forall t \in$ $\mathfrak{U}$ }.

Definition 1.1 ([11]): Let G be a non-empty set and κ ∈$\mathfrak{A}'$ with
||κ|| ≥ I. Suppose that a mapping S_b: G 3 →$\mathfrak{A}$ be a function satisfying the following properties
(S_{b_1}) :S_b (α, β, γ) ≽$0_{\mathfrak{A}}$for all α, β, γ ∈G,
(S_{b_2}): S_b (α, β, γ) = 0 ⇔ α = β = γ,
(S_{b_3}): S_b (α, β, γ) ≤ κ(S_b (α, α, θ) + S_b (β, β, θ) + S_b (γ, γ, θ)) for all α, β, γ, θ ∈G.
Then the function $S_b$is called a C ∗ -algebra valued S_b-metric on G and the pair (G,$\mathfrak{A}$, S_b) is called a C ∗ -algebra valued S_b-metric space (C ∗ -AV-S_bMS) with a co-efficient κ.

Example 1.1. Let G=R and $\mathfrak{A}$ =M2(R) be all 2×2 matrices with the usual operations of addition, scalar multiplication, and matrix multiplication. It is clear that ||P|| =$\sqrt{\sum_{i,j=1}^{2} |P_{i,j}|^2}$defines a norm on$\mathfrak{A}$, where P = (p_{ij}) ∈$\mathfrak{A}$. ∗: $\mathfrak{A}$ →$\mathfrak{A}$ defines an involution on $\mathfrak{A}$ and where $\mathfrak{A}$ ∗ =$\mathfrak{A}$. Then, $\mathfrak{A}$ is a C ∗ -algebra. For P = (p_{ij}) and Q = (q_{ij}) in$\mathfrak{A}$, a partial order on $\mathfrak{A}$ can be given as follows: P ≼ Q ⇐⇒$p_{ij} - q_{ij}$ ≤ 0 ∀i , j = 1, 2. Let (G, d) be a b-metric space where, ||κ|| ≥ I and Sb : G 3 → M2(R), fulfilling Sb(p, q, r) = $\begin{bmatrix} d(p, q) + d(q, r) + d(r, p) & 0 \\ 0 & d(p, q) + d(q, r) + d(r, p) \end{bmatrix}$. Then, clearly (G,$\mathfrak{A}$, Sb) is a C ∗ -AV-$S_b$ –MS.

Definition 1.2.([11]): AC∗-AV-S_bMS is symmetric if $S_b(\alpha,\beta,\gamma) = S_b(\alpha,\gamma,\beta)$=$S_b(\beta,\gamma,\alpha) = S_b(\beta,\alpha,\gamma) = S_b(\gamma,\alpha,\beta) = S_b(\gamma,\beta,\alpha)$∀ α, β,γ∈G

Definition 1.3. ([11]) : Let (G,$\mathfrak{A}$, S_b) be a C ∗ -AV-Sb-MS and $\{X_n\}$ be a sequence in G:
(1) If for all p ∈ N, ||S_b (X_{n+p}, X_{n+p}, X_n)|| → 0, where n → ∞, then $\{X_n\}$ is a Cauchy sequence in G.
(2) If ||S_b (X_n, X_n, χ)|| → 0, where n → ∞, then $\{X_n\}$ converges to χ, and we present it with $\lim_{n\to\infty} X_n = \chi$.
(3) If every Cauchy sequence is convergent in G , then (G,$\mathfrak{A}$, S_b) is a complete C ∗ -AV-Sb-MS.

Definition 1.4. ([11]): Suppose that ($\mathcal{G}_1$, $\mathfrak{A}_1$,S_{b_1}) and ($\mathcal{G}_2$, $\mathfrak{A}_2$,S_{b_2}) are C ∗ -AV-S_b MS, and let Γ: ($\mathcal{G}_1$, $\mathfrak{A}_1$,S_{b_1}) → ($\mathcal{G}_2$, $\mathfrak{A}_2$,S_{b_2}) be a function. Then, Γ is continuous at a point χ ∈$\mathcal{G}_1$if, for every sequence, $\{X_n\}$ in $\mathcal{G}_1$, $S_b(X_n, X_n, \chi)$ →$0_{\mathfrak{A}}$,(n → ∞) implies S_b (Γ(X_n), Γ(X_n), γ(χ))→$0_{\mathfrak{A}}$,where n → ∞.A function Γ is continuous at $\mathcal{G}_1$if and only if it is continuous at all χ ∈$\mathcal{G}_1$

Lemma 1.5. Suppose that $\mathfrak{A}$ is a unital C ∗ -algebra with a unit $1_{\mathfrak{A}}$: (1)
If $\{x_n\}_{n=1}^{\infty}$∞⊆$\mathfrak{A}$ and $\lim_{n\to\infty} x_n$ =$0_{\mathfrak{A}}$, then for any χ ∈$\mathfrak{A}$, $\lim_{n\to\infty}$χ ∗ χ_n χ=$0_{\mathfrak{A}}$. (2)
If χ, ξ ∈$\mathfrak{A}_h$and s ∈$\mathfrak{A}'_+$ then χ ≼ ξ yields sχ ≼sξ in which $\mathfrak{A}'_+$ = $\mathfrak{A}_+$∩ $\mathfrak{A}$ ′ . (3)
If χ ∈$\mathfrak{A}_+$with ||χ|| <$\frac{1}{2}$then $1_{\mathfrak{A}}$− χ is invertible, and ||χ ($1_{\mathfrak{A}}$− χ) −1 || < 1. (4)
If χ, ξ ∈$\mathfrak{A}_+$ such that χξ = ξχ, then χξ ≽$0_{\mathfrak{A}}$.

2. MAIN RESULTS

Definition 2.1: Let $(\mathcal{G},\mathfrak{U},S_b)$ be a C^*AVS_bMS with co-efficient$\|k\| > 1$.
Let Ω:$\mathcal{G}x\mathcal{G} \to \mathcal{G}$ be a mapping ,an element ($\varkappa$,α) ∈ $\mathcal{G}^2$ is called coupled fixed point of Ω if $\Omega(\varkappa,\alpha)$=$\varkappa$ *and* $\Omega(\alpha,\varkappa)$=α.
Definition 2.2: Let $(\mathcal{G},\mathfrak{U},S_b)$ be a C^*AVS_bMS with co-efficient$\|k\| > 1$ *and suppose* Ω:$\mathcal{G}x\mathcal{G} \to \mathcal{G}$ and $\Lambda : \mathcal{G} \to \mathcal{G}$ be two mappings:
(a) An element (ϰ,α) is siad to be a coupled coincident point of Ω and Λ if Ω(ϰ, α)=Λϰ ,Ω(α, ϰ)= Λα.
(b) element (ϰ,α) is siad to be common a coupled fixed point of Ω and Λ if Ω(ϰ, α)=Λϰ= ϰ,Ω(α, ϰ)= Λα= α.
(c) A pair of (Ω, Λ) is called weakly compitable if Λ(Ω(ϰ, α)) = Ω(Λϰ, Λα)) whenever for all ϰ, α ∈ $\mathcal{G}$ such that Ω(ϰ, α)=Λϰ ,Ω(α, ϰ)= Λα. In this Section we indicate;

$$\mathfrak{D} = \left\{ \begin{matrix} \varphi : \mathfrak{U}_+ \to \dfrac{\mathfrak{U}_+}{\varphi} \ \ is\ non-decreasing\ ,continuous\ and \\ \varphi(a) \prec a\ for\ all\ a \succ 0_{\mathfrak{U}}\ and\ \varphi(a) = 0_{\mathfrak{U}} \ \Leftrightarrow a = 0_{\mathfrak{U}} \end{matrix} \right\}$$

Lemma 2. 1. 1 : If $(\mathcal{G},\mathfrak{A},S_b)$ be a C^*AVS_bMS with $\|k\| \geq 1$ and suppose that $\{æ_p\}$ is a $C^* - AV{-}S_b$-convergent to ℓ , then we have $\frac{1}{2k}S_b(\wp,\wp,\ell) \preccurlyeq \liminf_{p\to\infty} S_b(\wp,\wp,æ_p) \preccurlyeq \limsup_{p\to\infty} S_b(\wp,\wp,æ_p) \preccurlyeq 2kS_b(\wp,\wp,\ell)$ for all $\wp \in \mathcal{G}$.
In particular, if $\ell = \wp$,then we have $\lim_{p\to\infty} S_b(\ell_p,\ell_p,\wp)$=$0_{\mathfrak{U}}$.
Proof : Using condition (S_{b_3}) of Definition 1.1, we have

$S_b(\wp,\wp,\text{æ}_p) \preccurlyeq 2kS_b(\wp,\wp,\ell) + kS_b(\text{æ}_p,\text{æ}_p,\ell)$ *and*
$S_b(\wp,\wp,\ell) \preccurlyeq 2kS_b(\wp,\wp,\text{æ}_p) + kS_b(\ell,\ell,\text{æ}_p)$
*Ta*king the upper limit as $p \to \infty$ in the first inequality and the lower limit as $p \to \infty$ in the second inequality we obtain the desired result.

Theorem 2.1.2: Let $(\mathcal{G},\mathfrak{A},S_b)$ be a complete C^*AVS_bMS with $\|k\| \geq 1$, suppose $\Gamma,\Omega : \mathcal{G}^2 \to \mathcal{G}$ and $\Lambda,\Theta : \mathcal{G} \to \mathcal{G}$ be four mappings satisfying for all $\ell,\varkappa,\text{æ},\text{œ} \in \mathcal{G}$,

$S_b(\Gamma(\ell,\varkappa),\Gamma(\ell,\varkappa),\Omega(\text{æ},\text{œ})) \preccurlyeq \frac{1}{4k^6}\varphi(a^*S_b(\Lambda\ell,\ \Lambda\ell,\Theta\text{æ})a + a^*S_b(\Lambda\varkappa,\ \Lambda\varkappa,\Theta\text{œ})a)$ ….. (2.1.1)

where $\varphi \in \mathfrak{D}$ and a$\in \mathfrak{A}$ in which $\|\sqrt{2}a\|<1$. Further assume that
$(i)\Gamma(\mathcal{G}^2) \subseteq \Theta(\mathcal{G})$ and $\Omega(\mathcal{G}^2) \subseteq \Lambda(\mathcal{G})$;
(ii) $\{\Gamma,\Lambda\}$ and $\{\Omega,\Theta\}$ are $\omega -$ compatible pairs;
(iii) one of $\Lambda(\mathcal{G})$ or $\Omega(\mathcal{G})$ is complete subspace of $(\mathcal{G},\mathfrak{A},S_b)$;
ThenΓ,Ω , Λ and Θ have a unique common coupled fixed point in $\mathcal{G}$.

proof: Let æ_0 ,$\text{œ}_0 \in \mathcal{G}$be arbitrary, and from (i), we construct the sequences $\{\text{æ}_p\}$, $\{\text{œ}_p\}$, in $\mathcal{G}$ as
$\Gamma(\text{æ}_{2p},\text{œ}_{2p}) = \Theta\text{æ}_{2p+1} = \ell_{2p}$, $\Gamma(\text{œ}_{2p},\text{æ}_{2p}) = \Theta\text{œ}_{2p+1} = \wp_{2p}$,
$\Omega(\text{æ}_{2p+1},\text{œ}_{2p+1}) = \Lambda\text{æ}_{2p+2} = \ell_{2p+1}$, $\Omega(\text{œ}_{2p+1},\text{æ}_{2p+1}) = \Lambda\text{œ}_{2p+2} = \wp_{2p+1}$ where p = 0,1,2 … … ….
Notices that in C^*-algebra, if a, b$\in \mathfrak{A}$ and a$\preccurlyeq b$, then for any j$\in \mathfrak{A}_+$ *both*j^*aj and j^*bj are positive elements and $j^*aj \preccurlyeq j^*bj$. Now we show that Γ,Ω , Λ and Θ have common coupled fixed point in $\mathcal{G}$. Assume that $S_b(\ell_{2p},\ell_{2p},\ell_{2p+1}) \succ 0_{\mathfrak{A}}$ and $S_b(\wp_{2p},\wp_{2p},\wp_{2p+1}) \succ 0_{\mathfrak{A}}$ $\forall\, p$ .Otherwise ,there exists some positive integer p such that $\ell_{2p} = \ell_{2p+1}$,$\wp_{2p} = \wp_{2p+1}$and so $(\ell_{2p},\wp_{2p})$ is a common coupled fixed point of Γ,Ω , Λ , Θ ,and the proof is complete .By using (2.1.1) ,for each p$\in N$,we have $S_b(\ell_{2p+1},\ell_{2p+1},\ell_{2p+2}) = S_b(\Gamma(\text{æ}_{2p+1},\text{œ}_{2p+1}),\Gamma(\text{æ}_{2p+1},\text{œ}_{2p+1}),\Omega(\text{æ}_{2p+2},\text{œ}_{2p+2}))$

$$\preccurlyeq \frac{1}{4k^6}\varphi(a^*S_b(\Lambda\text{æ}_{2p+1},\quad \Lambda\text{æ}_{2p+1},\Theta\text{æ}_{2p+2})a + a^*S_b(\Lambda\text{œ}_{2p+1},\quad \Lambda\text{œ}_{2p+1},\Theta\text{œ}_{2p+2})a)$$
$$\preccurlyeq \frac{1}{4k^6}\varphi(a^*S_b(\ell_{2p},\ell_{2p},\ell_{2p+1})a + a^*S_b(\wp_{2p},\wp_{2p},\wp_{2p+1})a) \ \ldots\ldots.. \quad (2.1.2)$$

and similarly, we prove that

$$S_b(\wp_{2p},\wp_{2p},\wp_{2p+1}) \preccurlyeq \frac{1}{4k^6}\varphi(a^*S_b(\ell_{2p},\ell_{2p},\ell_{2p+1})a + a^*S_b(\wp_{2p},\wp_{2p},\wp_{2p+1})a) \ \ldots. (2.1.3)$$

Let$\mathcal{L}_{2p+1} = S_b(\ell_{2p+1},\ell_{2p+1},\ell_{2p+2}) + S_b(\wp_{2p+1},\wp_{2p+1},\wp_{2p+2})$ and using eq. (2.1.2), (2.1,3), we have

$$\mathcal{L}_{2p+1} = S_b(\ell_{2p+1},\ell_{2p+1},\ell_{2p+2}) + S_b(\wp_{2p+1},\wp_{2p+1},\wp_{2p+2}) \prec \frac{1}{4k^6}\varphi(a^*S_b(\ell_{2p},\ell_{2p},\ell_{2p+1})a + a^*S_b(\wp_{2p},\wp_{2p},\wp_{2p+1})a)$$
$$+\frac{1}{4k^6}\varphi(a^*S_b(\ell_{2p},\ell_{2p},\ell_{2p+1})a + a^*S_b(\wp_{2p},\wp_{2p},\wp_{2p+1})a) \prec \frac{1}{4k^6}(\sqrt{2}a)^*\mathcal{L}_{2p}(\sqrt{2}a)$$
$$\prec \frac{1}{(4k^6)^2}[(\sqrt{2}a)^*]^2\mathcal{L}_{2p-1}(\sqrt{2}a)^2 \ldots \ldots\ldots\ldots \prec \frac{1}{(4k^6)^{2p+1}}[(\sqrt{2}a)^*]^{2p+1}\mathcal{L}_0(\sqrt{2}a)^{2p+1}$$

*N*ow, we can obtain for any p$\in N$

$$\mathcal{L}_p = S_b(\ell_p,\ell_p,\ell_{p+1}) + S_b(\wp_p,\wp_p,\wp_{p+1}) \prec \frac{1}{4k^6}(\sqrt{2}a)^*\mathcal{L}_{p-1}(\sqrt{2}a)\ldots\ldots\ldots \prec \frac{1}{(4k^6)^p}[(\sqrt{2}a)^*]^p\mathcal{L}_0(\sqrt{2}a)^p .$$

If $\mathfrak{I}_0 = 0_{\mathfrak{A}}$,then from (S_{b_3}) of Definition 1.3.1, we know $(\ell_0,\wp_0)$ is a coupled fixed point of Γ,Ω , Λ and Θ ,Now letting $\mathfrak{I}_0 \succ 0_{\mathfrak{A}}$,we get for any p$\in N$,for any $l \in N$ and using condition of (S_{b_3}) of Definition 1.3.1 ,

$$S_b(\ell_{2p+l},\ell_{2p+l},\ell_{2p}) \preccurlyeq k\begin{pmatrix} S_b(\ell_{2p+l},\ell_{2p+l},\ell_{2p+l-1}) + S_b(\ell_{2p+l},\ell_{2p+l},\ell_{2p+l-1}) \\ +S_b(\ell_{2p},\ell_{2p},\ell_{2p+l-1}) \end{pmatrix}$$
$$\preccurlyeq 2k\, S_b(\ell_{2p+l},\ell_{2p+l},\ell_{2p+l-1}) + kS_b(\ell_{2p},\ell_{2p},\ell_{2p+l-1})$$
$$= 2k\, S_b(\ell_{2p+l},\ell_{2p+l},\ell_{2p+l-1}) + kS_b(\ell_{2p+l-1},\ell_{2p+l-1},\ell_{2p})$$
$$\preccurlyeq 2k\, S_b(\ell_{2p+l},\ell_{2p+l},\ell_{2p+l-1}) + 2k^2S_b(\ell_{2p+l-1},\ell_{2p+l-1},\ell_{2p+l-2}) + k^2S_b(\ell_{2p+l-2},\ell_{2p+l-2},\ell_{2p}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\preccurlyeq 2k\, S_b(\ell_{2p+1},\ell_{2p+1},\ell_{2p+l-1})2k^2S_b(\ell_{2p+l-1},\ell_{2p+l-1},\ell_{2p+l-2}) + 2k^3S_b(\ell_{2p+l-2},\ell_{2p+l-2},\ell_{2p+l-3}) +$$
$$+2k^lS_b(\ell_{2p+1},\ell_{2p+1},\ell_{2p})\ \textit{and}\ \text{ similarly,}$$

$S_b(\wp_{2p+1},\wp_{2p+1},\wp_{2p}) \preccurlyeq 2k\, S_b(\wp_{2p+l},\wp_{2p+l},\wp_{2p+l-1}) + 2k^2S_b(\wp_{2p+l-1},\wp_{2p+l-1},\wp_{2p+l-2}) +$
$2k^3S_b(\wp_{2p+l-2},\wp_{2p+l-2},\wp_{2p+l-3}) + \ldots\ldots\ldots + 2k^lS_b(\wp_{2p+1},\wp_{2p+1},\wp_{2p})$ Consequently,
$S_b(\ell_{2p+l},\ell_{2p+l},\ell_{2p}) + S_b(\wp_{2p+l},\wp_{2p+l},\wp_{2p}) \preccurlyeq 2k\mathcal{L}_{2p+l-1} + 2k^2\mathcal{L}_{2p+l-2} + \ldots. + 2k^l\mathcal{L}_{2p}$
$\prec \frac{2k}{(4k^6)^{2p+l-1}}[(\sqrt{2}a)^*]^{2p+l-1}\mathcal{L}_0(\sqrt{2}a)^{2p+l-1} + \frac{2k^2}{(4k^6)^{2p+l-2}}[(\sqrt{2}a)^*]^{2p+l-2}\mathcal{L}_0(\sqrt{2}a)^{2p+l-2} + \ldots\ldots. +$
$\frac{2k^l}{(4k^6)^{2p}}[(\sqrt{2}a)^*]^{2p}\mathcal{L}_0(\sqrt{2}a)^{2p} \preccurlyeq 2\sum_{m=2p}^{2p+l-1}\frac{k^{l-m+2p}}{(4k^6)^m}\left(((\sqrt{2}\,a)^*)^m\mathcal{L}_0(\sqrt{2}a)^m\right)$
$= 2\sum_{m=2p}^{2p+l-1}\left(\left((\sqrt{2}\,a)^*\right)^m\left(\frac{k^{l-m+2p}}{(4k^6)^m}\right)^{\frac{1}{2}}\mathcal{L}_0^{\frac{1}{2}}\right)\left(\mathcal{L}_0^{\frac{1}{2}}\left(\frac{k^{l-m+2p}}{(4k^6)^m}\right)^{\frac{1}{2}}(\sqrt{2}a)^m\right)$
$= 2\sum_{m=2p}^{2p+l-1}\left((\sqrt{2}a)^m\left(\frac{k^{l-m+2p}}{(4k^6)^m}\right)^{\frac{1}{2}}\mathcal{L}_0^{\frac{1}{2}}\right)^*\left(\mathcal{L}_0^{\frac{1}{2}}\left(\frac{k^{l-m+2p}}{(4k^6)^m}\right)^{\frac{1}{2}}(\sqrt{2}a)^m\right)$
$\leq 2\sum_{m=2p}^{2p+l-1}\left\|\mathcal{L}_0^{\frac{1}{2}}\left(\frac{k^{l-m+2p}}{(4k^6)^m}\right)^{\frac{1}{2}}(\sqrt{2}a)^m\right\|^2 1_{\mathfrak{A}} \ \leq 2\left\|\mathcal{L}_0^{\frac{1}{2}}\right\|\sum_{m=2p}^{2p+l-1}\|\sqrt{2}a\|^m\left\|\frac{k^{l-m+2p}}{(4k^6)^m}\right\| 1_{\mathfrak{A}}$
$\leq 2\,\|\mathcal{L}_0\|\frac{\|k\|^{l+1}\|\sqrt{2}a\|^{2p}}{\|4k^6\|^{2p-1}(\|4k^7\|-\|\sqrt{2}a\|)}1_{\mathfrak{A}} \ \to 0$ as p$\to \infty$

in which $1_{\mathfrak{U}}$ is the unit element in $\mathfrak{U}$ and together with
$S_b(\ell_{2p+l},\ell_{2p+l},\ell_{2p}) \leq S_b(\ell_{2p+l},\ell_{2p+l},\ell_{2p}) + S_b(\wp_{2p+l},\wp_{2p+l},\wp_{2p})$ and
$S_b(\wp_{2p+l},\wp_{2p+l},\wp_{2p}) \leq S_b(\ell_{2p+l},\ell_{2p+l},\ell_{2p}) + S_b(\wp_{2p+l},\wp_{2p+l},\wp_{2p})$ implies that $\{\ell_{2p}\}$,$\{\wp_{2p}\}$ are Cauchy sequence in $\mathcal{G}$ with respect to $1_{\mathfrak{U}}$. It follows that $\{\ell_{2p+1}\}$,$\{\wp_{2p+1}\}$ are Cauchy sequence in $\mathcal{G}$ with respect to $1_{\mathfrak{U}}$ and hence $\{\ell_{2p}\}$,$\{\wp_{2p}\}$ are Cauchy sequence in $(\mathcal{G},\mathfrak{A},S_b)$ .Suppose $\Lambda(\mathcal{G})$ is complete subspace of $(\mathcal{G},\mathfrak{A},S_b)$,then the sequences $\{\ell_p\}$,$\{\wp_p\}$ are converge to æ and œ respectively in $\Lambda(\mathcal{G})$ .Thus there exist $\ell,\wp \in \Lambda(\mathcal{G})$ such that
$\lim_{p\to\infty} \ell_{2p} = \lim\Lambda_{p\to\infty} \text{æ}_{2p+1} = \text{æ} = \Lambda\ell$ and $\lim_{p\to\infty} \wp_{2p} = \lim\Lambda_{p\to\infty} \text{œ}_{2p+1} = \text{œ} = \Lambda\wp$ …… (2.1.4)
Now we show that $\Gamma(\ell,\wp) = \text{æ}$and $\Gamma(\wp,\ell) = \text{œ}$ Suppose $\Gamma(\ell,\wp) \neq \text{æ}$and $\Gamma(\wp,\ell) \neq \text{œ}$ by Lemma (5.1.1), we have

$$\frac{1}{2k}S_b(\Gamma(\ell,\wp),\Gamma(\ell,\wp),\text{æ}) \preccurlyeq \liminf_{p\to\infty} S_b(\Gamma(\ell,\wp),\Gamma(\ell,\wp),\ell_{2p+1})$$

$$=\liminf_{p\to\infty} S_b(\Gamma(\ell,\wp),\Gamma(\ell,\wp),\Gamma(\text{æ}_{2p+1},\text{œ}_{2p+1}))$$

$$\preccurlyeq \limsup_{p\to\infty} \frac{1}{4k^6}\varphi(a^*S_b(\Lambda\ell, \quad \Lambda\ell,\Theta\text{æ}_{2p+1})a + a^*S_b(\Lambda\wp, \quad \Lambda\wp,\Theta\text{œ}_{2p+1})a)$$

$$\preccurlyeq \limsup_{p\to\infty} \frac{1}{4k^6}\varphi(a^*S_b(\Lambda\ell,\Lambda\ell,\ell_{2p})a + a^*S_b(\Lambda\wp,\Lambda\wp,\wp_{2p})a)$$

$$\prec \limsup_{p\to\infty}(a^*S_b(\Lambda\ell,\Lambda\ell,\ell_{2p})a + a^*S_b(\Lambda\wp,\Lambda\wp,\wp_{2p})a) = 0_{\mathfrak{U}} .$$

Hence , we obtain that $S_b(\Gamma(\ell,\wp),\Gamma(\ell,\wp),\text{æ}) = 0_{\mathfrak{U}}$ implies $\Gamma(\ell,\wp) = \text{æ}$and similarly we get $S_b(\Gamma(\wp,\ell),\Gamma(\wp,\ell),\text{œ}) = 0_{\mathfrak{U}}$ so that $\Gamma(\wp,\ell) = \text{œ}$. It follows that $\Gamma(\ell,\wp) = \text{æ} = \Lambda\ell$ and $\Gamma(\wp,\ell) = \text{œ} = \Lambda\wp$ .Since $\{\Gamma,\Lambda\}$ is weakly compatible pair, we have $\Gamma(\text{æ},\text{œ}) = \Lambda\text{æ}$ *and* $\Gamma(\text{œ},\text{æ}) = \Lambda\text{œ}$, then we prove that $\Lambda\text{æ} = \text{æ}$and $\Lambda\text{œ} = \text{œ}$ .From Lemma (2.1.1)
we have

$$\frac{1}{k^2}S_b(\Lambda\text{æ},\Lambda\text{æ},\ell_{2p+1}) \preccurlyeq \liminf_{p\to\infty} S_b(\Gamma(\text{æ},\text{œ}),\Gamma(\text{æ},\text{œ}),\Gamma(\text{æ}_{2p+1},\text{œ}_{2p+1}))$$

$$\preccurlyeq \limsup_{p\to\infty} \frac{1}{4k^6}\varphi(a^*S_b(\Lambda\text{æ},\Lambda\text{æ},\Theta\text{æ}_{2p+1})a + a^*S_b(\Lambda\text{œ},\Lambda\text{œ},\Theta\text{œ}_{2p+1})a)$$

$$\preccurlyeq \limsup_{p\to\infty} \frac{1}{4k^4}\varphi(k^2(a^*S_b(\Lambda\text{æ},\Lambda\text{æ},\ell_{2p})a + a^*S_b(\Lambda\text{œ},\Lambda\text{œ},\wp_{2p})a))$$

And from Lemma (5.1,1) and using above inequality, we have

$$\frac{1}{2k}S_b(\Lambda\text{æ},\Lambda\text{æ},\text{æ}) \preccurlyeq \liminf_{p\to\infty} S_b(\Lambda\text{æ},\Lambda\text{æ},\ell_{2p+1})$$

$$\preccurlyeq \limsup_{p\to\infty} \frac{1}{4k^4}\varphi(k^2(a^*S_b(\Lambda\text{æ},\Lambda\text{æ},\ell_{2p})a + a^*S_b(\Lambda\text{œ},\Lambda\text{œ},\wp_{2p})a))$$

$$\preccurlyeq \limsup_{p\to\infty} \frac{1}{4k^4}\varphi(2k^3(a^*S_b(\Lambda\text{æ},\Lambda\text{æ},\text{æ})a + a^*S_b(\Lambda\text{œ},\Lambda\text{œ},\text{œ})a))$$

$$\prec \frac{1}{2k}(a^*S_b(\Lambda\text{æ},\Lambda\text{æ},\text{æ})a + a^*S_b(\Lambda\text{œ},\Lambda\text{œ},\text{œ})a)$$

And similarly,

$$\frac{1}{2k}S_b(\Lambda\text{œ},\Lambda\text{œ},\text{œ}) \prec \frac{1}{2k}(a^*S_b(\Lambda\text{æ},\Lambda\text{æ},\text{æ})a + a^*S_b(\Lambda\text{œ},\Lambda\text{œ},\text{œ})a)$$

Therefore ,
$S_b(\Lambda\text{æ},\Lambda\text{æ},\text{æ}) + S_b(\Lambda\text{œ},\Lambda\text{œ},\text{œ}) \prec (\sqrt{2}\,a)^*(S_b(\Lambda\text{æ},\Lambda\text{æ},\text{æ}) + S_b(\Lambda\text{œ},\Lambda\text{œ},\text{œ}))(\sqrt{2}\,a)$.
It follows that,
$\|S_b(\Lambda\text{æ},\Lambda\text{æ},\text{æ}) + S_b(\Lambda\text{œ},\Lambda\text{œ},\text{œ})\| \prec 2\|a\|^2\|S_b(\Lambda\text{æ},\Lambda\text{æ},\text{æ}) + S_b(\Lambda\text{œ},\Lambda\text{œ},\text{œ})\|$
Since$\|a\| < \frac{1}{\sqrt{2}}$, which implies that $\|S_b(\Lambda\text{æ},\Lambda\text{æ},\text{æ}) + S_b(\Lambda\text{œ},\Lambda\text{œ},\text{œ})\| = 0$ and
hence$S_b(\Lambda\text{æ},\Lambda\text{æ},\text{æ}) = 0_{\mathfrak{U}}$,$S_b(\Lambda\text{œ},\Lambda\text{œ},\text{œ}) = 0_{\mathfrak{U}}$ imply that $\Lambda\text{æ} = \text{æ}$,$\Lambda\text{œ} = \text{œ}$.
Thus $(\text{æ},\text{œ})$ is common coupled fixed point of Γ and Λ . Since $\Gamma(\mathcal{G}^2) \subseteq \Theta(\mathcal{G})$ so there exist $\varkappa,\mathfrak{B} \in \mathcal{G}$ such that $\Gamma(\text{æ},\text{œ}) = \text{æ} = \Theta\varkappa$ and $\Gamma(\text{œ},\text{æ}) = \text{œ} = \Theta\mathfrak{B}$. Now show that $\Omega(\varkappa,\mathfrak{B}) = \text{æ}$ and $\Omega(\mathfrak{B},\varkappa) = \text{œ}$ .Now from (5.1.1), we have
$0_{\mathfrak{U}} \preccurlyeq S_b(\text{æ},\text{æ},\Omega(\varkappa,\mathfrak{B})) = S_b(\Gamma(\text{æ},\text{œ}),\Gamma(\text{æ},\text{œ}),\Omega(\varkappa,\mathfrak{B})) \preccurlyeq \frac{1}{4k^6}\varphi(a^*S_b(\Lambda\text{æ},\Lambda\text{æ},\Theta\varkappa)a + a^*S_b(\Lambda\text{œ},\Lambda\text{œ},\Theta\mathfrak{B})a)$

$$\prec a^*S_b(\Lambda\text{æ},\Lambda\text{æ},\Theta\varkappa)a + a^*S_b(\Lambda\text{œ},\Lambda\text{œ},\Theta\mathfrak{B})a = 0_{\mathfrak{U}} .$$

Which implies that $\Omega(\varkappa,\mathfrak{B}) = \text{æ}$ and similarly, we get that $\Omega(\mathfrak{B},\varkappa) = \text{œ}$. Since $\{\Omega,\Theta\}$ is weakly compatible pair, we have $\Omega(\text{æ},\text{œ}) = \Theta\text{æ}$ and $\Omega(\text{œ},\text{æ}) = \Theta\text{œ}$,then we prove that $\Theta\text{æ} = \text{æ}$ and $\Theta\text{œ} = \text{œ}$ .Now from (5.1.1), we have
$0_{\mathfrak{U}} \preccurlyeq S_b(\text{æ},\text{æ},\Theta\text{æ}) = S_b(\Gamma(\text{æ},\text{œ}),\Gamma(\text{æ},\text{œ}),\Omega(\text{æ},\text{œ}))$

$$\preccurlyeq \frac{1}{4k^6}\varphi(a^*S_b(\Lambda\text{æ},\Lambda\text{æ},\Theta\text{æ})a + a^*S_b(\Lambda\text{œ},\Lambda\text{œ},\Theta\text{œ})a)$$

$$< a^*S_b(\text{æ},\text{æ},\Theta\text{æ})a + a^*S_b(\text{œ},\text{œ},\Theta\text{œ})a$$

Similarly,

$$0_{\mathfrak{U}} \preccurlyeq S_b(\text{œ},\text{œ},\Theta\text{œ}) < a^*S_b(\text{æ},\text{æ},\Theta\text{æ})a + a^*S_b(\text{œ},\text{œ},\Theta\text{œ})a$$

Therefore,
$0_{\mathfrak{U}} \preccurlyeq S_b(\text{æ},\text{æ},\Theta\text{æ}) + S_b(\text{œ},\text{œ},\Theta\text{œ}) \prec (\sqrt{2}\,a)^*(S_b(\text{æ},\text{æ},\Theta\text{æ}) + S_b(\text{œ},\text{œ},\Theta\text{œ}))(\sqrt{2}\,a)$.

which implies that

$0 \leq \|S_b(\text{æ},\text{æ},\Theta\text{æ}) + S_b(\text{œ},\text{œ},\Theta\text{œ})\| \prec 2\|a\|^2\|S_b(\text{æ},\text{æ},\Theta\text{æ}) + S_b(\text{œ},\text{œ},\Theta\text{œ})\|$

$Since\|a\| < \frac{1}{\sqrt{2}}$, which implies that $\|S_b(\text{æ},\text{æ},\Theta\text{æ}) + S_b(\text{œ},\text{œ},\Theta\text{œ})\| = 0$ and

$hence S_b(\text{æ},\text{æ},\Theta\text{æ}) = 0_{\mathfrak{A}}$,$S_b(\text{œ},\text{œ},\Theta\text{œ}) = 0_{\mathfrak{A}}$ imply that $\Theta\text{æ} = \text{æ}$,$\Theta\text{œ} = \text{œ}$.

$Hence \Omega(\text{æ},\text{œ}) = \Theta\text{æ} = \text{æ}$ and $\Omega(\text{œ},\text{æ}) = \Theta\text{œ} = \text{œ}$. Thus $(\text{æ},\text{œ})$ is common coupled fixed point of Γ,Ω , Λ and Θ . In the following we will show the uniqueness of common coupled fixed point in $\mathcal{G}$. Let us take $(\text{æ},\text{œ}) \neq (\text{æ}',\text{œ}')$ be another fixed point of Γ,Ω , Λ and Θ then ,

$$0_{\mathfrak{A}} \preccurlyeq S_b(\text{æ},\text{æ},\text{æ}') = S_b(\Gamma(\text{æ},\text{œ}),\Gamma(\text{æ},\text{œ}),\Omega(\text{æ}',\text{œ}')) \preccurlyeq \frac{1}{4k^6}\varphi(a^*S_b(\Lambda\text{æ},\Lambda\text{æ},\Theta\text{æ}')a + a^*S_b(\Lambda\text{œ},\Lambda\text{œ},\Theta\text{œ}')a)$$

$$\prec a^*S_b(\text{æ},\text{æ},\text{æ}')a + a^*S_b(\text{œ},\text{œ},\text{œ}')a \quad similarly ,$$

$$0_{\mathfrak{A}} \preccurlyeq S_b(\text{œ},\text{œ},\text{œ}') \prec a^*S_b(\text{æ},\text{æ},\text{æ}')a + a^*S_b(\text{œ},\text{œ},\text{œ}')a$$

Therefore,

$0 \leq \|S_b(\text{æ},\text{æ},\text{æ}') + S_b(\text{œ},\text{œ},\text{œ}')\| \prec 2\|a\|^2\|S_b(\text{æ},\text{æ},\text{æ}') + S_b(\text{œ},\text{œ},\text{œ}')\|$

since $\|\sqrt{2}\,a\|$<1, it is incongruous. Consequently $\text{æ} = \text{æ}'$ and $\text{œ} = \text{œ}'$ .Therefore , the UCCFP of Γ,Ω , Λ and Θ is $(\text{æ},\text{œ})$. In order to prove that Γ,Ω , Λ and Θ have a unique fixed point, we only have to prove $\text{æ} = \text{œ}$,we have

$$0_{\mathfrak{A}} \preccurlyeq S_b(\text{æ},\text{æ},\text{œ}) = S_b(\Gamma(\text{æ},\text{œ}),\Gamma(\text{æ},\text{œ}),\Omega(\text{œ},\text{æ})) \preccurlyeq \frac{1}{4k^6}\varphi(a^*S_b(\Lambda\text{æ},\Lambda\text{æ},\Theta\text{œ})a + a^*S_b(\Lambda\text{œ},\Lambda\text{œ},\Theta\text{æ})a)$$

$$\prec a^*S_b(\text{æ},\text{æ},\text{œ})a + a^*S_b(\text{œ},\text{œ},\text{æ})a \text{ ………. } (2.1.5)$$

Therefore, from definition of 1.3.2, we get that $0 \leq \|S_b(\text{æ},\text{æ},\text{œ})\| \leq 2\|a\|^2\|S_b(\text{æ},\text{æ},\text{œ}) + S_b(\text{æ},\text{æ},\text{œ})\| < \|S_b(\text{æ},\text{æ},\text{œ})\|$ This is incongruous. Consequently, $\text{æ} = \text{œ}$,which means that Γ,Ω , Λ and Θ have a unique fixed point of the form $(\text{æ},\text{æ})$ in $\mathcal{G}$.

Theorem 2.1.3: Let $(\mathcal{G},\mathfrak{A},S_b)$ be a complete C^*-AV-S_bMS , Suppose $\Gamma : \mathcal{G}^2 \rightarrow \mathcal{G}$ and $\Lambda: \mathcal{G} \rightarrow \mathcal{G}$ be two mappings with the following assumptions;

(i)$\Gamma(\mathcal{G}^2) \subseteq \Lambda(\mathcal{G})$ And $\Lambda(\mathcal{G})$ is a closed sub space of $\mathcal{G}$;

$(ii) S_b(\Gamma(\ell,\varkappa),\Gamma(\ell,\varkappa),\Omega(\text{æ},\text{œ})) \preccurlyeq \frac{1}{4k^6}\varphi(a^*S_b(\Lambda\ell,\Lambda\ell,\Lambda\text{æ})a + a^*S_b(\Lambda\varkappa,\Lambda\varkappa,\Lambda\text{œ})a) where\ \varphi \in \mathfrak{D}$ and a$\in \mathfrak{A}$ in which $\|\sqrt{2}a\|$<1.

(iii) $\{ \Gamma , \Lambda \}$ is ω-compatible pairs.

Then Γ and Λ have a unique common coupled fixed point in $\mathcal{G}$.

Proof : The proof follows from Theorem (2.1.2) by taking $\Gamma = \Omega$ and $\Lambda = \Theta$.

***Corollary* 2.1.4.** Let $(\mathcal{G},\mathfrak{A},S_b)$ be a complete C^*-AV-S_bMS , $\Gamma : \mathcal{G}^2 \rightarrow \mathcal{G}$ is satisfying $S_b(\Gamma(\ell,\varkappa),\Gamma(\ell,\varkappa),\Gamma(\text{æ},\text{œ})) \preccurlyeq \frac{1}{4k^6}\varphi(a^*S_b(\ell,\ell,\text{æ})a + a^*S_b(\varkappa,\varkappa,\text{œ})a) where\ \varphi \in \mathfrak{D}$ and a$\in \mathfrak{A}$ in which $\|\sqrt{2}a\|$<1. Then Γ has a unique coupled fixed point in $\mathcal{G}$.

Proof : The proof follows from Theorem (2.1.2) by taking $\Gamma = \Omega$ and $\Lambda = \Theta = I_{\mathcal{G}}$.

***Example* 2.1.** Let $\mathcal{G}$ =[0,1] and $\mathfrak{A} = M_2(\mathbb{R})$ be a all 2x2 matrices whose norm is $\|\mathfrak{A}\|$=max $\{a_1,a_2,a_3,a_4\}$ where a_i's are the entries of $\mathfrak{A}$ .Then ,clearly $(\mathcal{G},\mathfrak{A},S_b)$ be a complete C^*-AV-S_bMS with k=4 whenever $S_b: \mathcal{G}^3 \rightarrow M_2(\mathbb{R})$ be as $S_b(p,q,r) = ((|q + r - 2p| + |q - r|)^2 , 0)$. Let $\varphi: \mathfrak{A}_+ \rightarrow \mathfrak{A}_+$ defined by $\varphi(\aleph) = \frac{\aleph}{8}$,and a$\in \mathfrak{A}$ with $\|a\| < \frac{1}{\sqrt{2}} < 1$. we define mappings Γ,Ω $: \mathcal{G}^2 \rightarrow \mathcal{G}$,$\Lambda,\Theta : \mathcal{G} \rightarrow \mathcal{G}$ *as follows* $\Gamma(\text{æ},\text{œ}) = \frac{\text{æ}^2+\text{œ}^2}{4^6}$, $\Omega(\text{æ},\text{œ}) = \frac{\text{æ}^2+\text{œ}^2}{4^7}$,$\Lambda(\text{æ}) = \frac{\text{æ}^2}{4}$ and $\Theta(\text{æ}) = \frac{\text{æ}^2}{16}$**.** Then clearly,$\Gamma(\mathcal{G}^2) \subseteq \Theta(\mathcal{G})$ and $\Omega(\mathcal{G}^2) \subseteq \Lambda(\mathcal{G})$. One can show that $(\text{æ},\text{œ})$ is a coupled coincident point of Γ,Ω , Λ and Θ if and only if $\text{æ} = \text{œ}$=0. Since $\Gamma(\Lambda 0,\Lambda 0) = \Lambda(\Gamma(0,0))$ and $\Omega(\Theta 0,\Theta 0) = \Theta(\Omega(0,0))$,we get that

$\{ \Gamma,\Lambda\}$ and $\{ \Omega,\Theta\}$ are ω-compatible pairs. Now from inequality (2.1.1), we have

$$S_b(\Gamma(\ell,\varkappa),\Gamma(\ell,\varkappa),\Omega(\text{æ},\text{œ})) = \left(\begin{pmatrix}|\Gamma(\ell,\varkappa) + \Omega(\text{æ},\text{œ}) - 2\Gamma(\ell,\varkappa)| \\ +|\Gamma(\ell,\varkappa) - \Omega(\text{æ},\text{œ})|\end{pmatrix}^2 \quad 0\right)$$

$$= \left(\begin{pmatrix}\left|\frac{\text{æ}^2+\text{œ}^2}{4^7} - \frac{\ell^2+\varkappa^2}{4^6}\right| \\ + \left|\frac{\ell^2+\varkappa^2}{4^6} - \frac{\text{æ}^2+\text{œ}^2}{4^7}\right|\end{pmatrix}^2 \quad 0\right)$$

$$= \left(4\left(\left|\frac{\ell^2+\varkappa^2}{4^6} - \frac{\text{æ}^2+\text{œ}^2}{4^7}\right|\right)^2 \quad 0\right)$$

$$= \left(4\left(\left|\frac{4\ell^2-\text{æ}^2+4\varkappa^2-\text{œ}^2}{4^7}\right|\right)^2 \quad 0\right)$$

$$= 4\frac{1}{(4^5)^2}\left(\left|\frac{4\ell^2-\text{æ}^2+4\varkappa^2-\text{œ}^2}{16}\right|^2 ,0\right)$$

$$\preccurlyeq 4\frac{1}{(4^5)^2}\left(2\left|\frac{4\ell^2-\text{æ}^2}{16}\right|^2 + 2\left|\frac{4\varkappa^2-\text{œ}^2}{16}\right|^2 ,0\right)$$

$$\preccurlyeq \frac{1}{4(4^6)}\frac{1}{8}\begin{pmatrix} \left(\frac{1}{\sqrt{2}},0\right)\begin{pmatrix}2\left|\frac{4\ell^2-\text{æ}^2}{16}\right|^2 & 0\end{pmatrix}\left(\frac{1}{\sqrt{2}},0\right) \\ +\left(\frac{1}{\sqrt{2}},0\right)\begin{pmatrix}2\left|\frac{4\varkappa^2-\text{œ}^2}{16}\right|^2 & 0\end{pmatrix}\left(\frac{1}{\sqrt{2}},0\right)\end{pmatrix}$$

$$\preccurlyeq \frac{1}{4k^6}\varphi(a^*S_b(\Lambda\ell,\Lambda\ell,\Lambda\text{æ})a + a^*S_b(\Lambda\varkappa,\Lambda\varkappa,\Lambda\text{œ})a)$$

where a=$\begin{bmatrix}\frac{1}{\sqrt{2}} & 0\\ 0 & \frac{1}{\sqrt{2}}\end{bmatrix}$ with $\|a\| = \frac{1}{\sqrt{2}}$<1. Thus, all conditions of Theorem 2.1.2 are satisfied and therefore Γ,Ω , Λ and Θ have a unique common fixed point (namely $(\text{æ},\text{æ}) = (0,0)$) in $\mathcal{G}$.

2.2. APPLICATION TO INTEGRAL EQUATIONS

As an application to corollary 2.1.4. we examine the existence of a singular solution to an integral equation in this section.

Theorem2.2.1: Consider the integral equation

$\ell(t) = \int_\varepsilon (\mathfrak{f}(t,s,\ell(s)) + \mathfrak{g}(t,s,\ell(s)))ds\ \forall$ t∈ ε ………………………. (2.1.6)

where $\varepsilon$is a Lebesgue measurable set and m (ε)<∞ . Suppose that

$(i_0)\mathfrak{f},\mathfrak{g} :\varepsilon \times \varepsilon \times \mathbb{R} \to \mathbb{R}$ are integrable,

(i_1) there exists a continuous function $\phi :\varepsilon \times \varepsilon \to \mathbb{R}$ and $\theta \in (0,1)$ such that $i,j \in \varepsilon$ and $\ell(i)$,$\varkappa(j) \in \mathbb{R}$,

$\left|\mathfrak{f}(i,j,\ell(j)) - \mathfrak{f}(i,j,\varkappa(j))\right| \le \frac{\theta^{\frac{1}{p}}}{4k^3}|\phi(i,j)||\ell(j)-\varkappa(j)|$ and $\left|\mathfrak{g}(i,j,\ell(j)) - \mathfrak{g}(i,j,\varkappa(j))\right| \le \frac{\theta^{\frac{1}{p}}}{4k^3}|\phi(i,j)||\ell(j)-\varkappa(j)|$

$(i_2) sup_{i\in\varepsilon}\int_\varepsilon |\phi(i,j)|\,dj \le 1$then the integral equation has a unique solution in $L^\infty(\varepsilon)$.

Proof: Let $\varepsilon = [0,1]$ and $\mathcal{G} = L^\infty(\varepsilon)$ be the set of essential bounded measurable function on ε and $\mathcal{H}=L^2(\varepsilon)$ .The set of bounded linear operators on Hilbert space $\mathcal{H}$ denoted by L $(\mathcal{H})$. we equip $\mathcal{G}$ with $S_b : \mathcal{G}\times\mathcal{G}\times\mathcal{G} \to \mathrm{L}(\mathcal{H})$,which is ascertained by $S_b(\text{æ},\mathfrak{B},\text{œ}) = M_{(|\text{æ}-\text{œ}|+|\mathfrak{B}-\text{œ}|)^p}$,where $M_{(|\text{æ}-\text{œ}|+|\mathfrak{B}-\text{œ}|)^p}$ is the multiplication operator on $\mathcal{H}$ ascertained by $M_h(\propto) = h\propto$,$\propto\in\mathcal{H}$ .Therefore , $(\mathcal{G},\mathrm{L}(\mathcal{H}),S_b)$ is a complete C^*-AV-S_bMS with k=$2^{2(p-1)}$ where p=2>1 .Define the mappings $\psi:\mathfrak{U}_+ \longrightarrow \mathfrak{U}_+$by $\psi(a) = \frac{a}{2},\phi(a) = \frac{2a}{3}$ and $\Gamma:\mathcal{G}^2 \longrightarrow \mathcal{G}$ as for all $t\in\varepsilon$. $\Gamma(\ell,\varkappa)(t) = \int_\varepsilon (\mathfrak{f}(t,s,\ell(s)) + \mathfrak{g}(t,s,\ell(s)))ds\ \forall t\in\varepsilon$ we have$S_b(\Gamma(\ell,\varkappa),\Gamma(\ell,\varkappa),\Gamma(\text{æ},\text{œ})) = M_{(2|\Gamma(\ell,\ \varkappa)-\Gamma(\text{æ},\text{œ})|)^p}$

Let us first evaluate the following expression:

$$2^p|\Gamma(\ell,\varkappa)-\Gamma(\text{æ},\text{œ})(t)|^p$$

$$=2^p\left|\int_\varepsilon (\mathfrak{f}(t,s,\ell(s)) + \mathfrak{g}(t,s,\varkappa(s)))ds - \int_\varepsilon (\mathfrak{f}(t,s,\text{æ}(s)) + \mathfrak{g}(t,s,\text{œ}(s)))ds\right|^p$$

$$=2^p\left|\int_\varepsilon (\mathfrak{f}(t,s,\ell(s)) - \mathfrak{f}(t,s,\text{æ}(s)))ds + \int_\varepsilon (\mathfrak{g}(t,s,\varkappa(s)) - \mathfrak{g}(t,s,\text{œ}(s)))ds\right|^p$$

$$\preccurlyeq 2^p2^{p-1}\left(\int_\varepsilon \frac{\theta^{\frac{1}{p}}}{4k^3}|\phi(t,s)|\,ds\right)^p(|\ell(s)-\text{æ}(s)|^p + |\varkappa(s)-\text{œ}(s)|^p)$$

$$\preccurlyeq \frac{\theta}{2(2k^3)^p}\left(sup_{i\in\varepsilon}\int_\varepsilon |\phi(t,s)|\,ds\right)^p[(2\|\ell(s)-\text{æ}(s)\|_\infty) + (2\|\varkappa(s)-\text{œ}(s)\|_\infty))^p]$$

$$\preccurlyeq \frac{\theta}{2(2k^3)^p}[(2\|\ell(s)-\text{æ}(s)\|_\infty) + (2\|\varkappa(s)-\text{œ}(s)\|_\infty))^p]$$

Therefore,

$$\|S_b(\Gamma(\ell,\varkappa),\Gamma(\ell,\varkappa),\Gamma(\text{æ},\text{œ}))\| = sup_{\|h\|=1} < M_{(2|\Gamma(\ell,\ \varkappa)-\Gamma(\text{æ},\text{œ})|)^p}h,\ h >$$

$$= sup_{\|h\|=1} < 2^pM_{(|\Gamma(\ell,\ \varkappa)-\Gamma(\text{æ},\text{œ})|)^p}h,\ h >$$

$$=sup_{\|h\|=1}\int_\varepsilon (2^p|\Gamma(\ell,\ \varkappa)(t)-\Gamma(\text{æ},\text{œ})(t)|^p)\,h(t)\overline{h(t)}\,\mathrm{dt}$$

$$\le sup_{\|h\|=1}\int_\varepsilon |h(t)|^2dt\,\frac{\theta}{2(2k^3)^p}[(2\|\ell(s)-\text{æ}(s)\|_\infty) + (2\|\varkappa(s)-\text{œ}(s)\|_\infty))^p]$$

$$\le \frac{\theta}{2(2k^3)^p}[\ \|S_b(\ell,\ell,\text{æ})\| + \|S_b(\varkappa,\varkappa,\text{œ})\|]$$

$$\le \left\|\frac{1}{4k^6}\varphi\left(a^*S_b(\ell,\ \ell,\text{æ})a + a^*S_b\left(\varkappa,\ \varkappa,\text{œ}\right)a\right)\right\|$$

By setting a=$\sqrt{\theta}1_{\mathrm{L}(\mathcal{H})}$,then a∈ $\mathrm{L}(\mathcal{H})$ and $\|a\| = \sqrt{\theta}<\frac{1}{\sqrt{2}}$ .Hence , applying our Corollary 2.1.4, we get desired result.

2.3 APPLICATION TO HOMOTOPY

This section explores the possibility of a unique solution to the Homotopy Hypothesis

Theorem2.3.1: If$(\mathcal{G},\mathfrak{U},S_b)$ be a complete C^*-AV-S_bMS ,then $\mathfrak{U}$ and $\overline{\mathfrak{U}}$ are open and closed subsets of $\mathcal{G}$,respectively , such that $\mathfrak{U}\subseteq\overline{\mathfrak{U}}$ .Let $\mathfrak{L}_b:\overline{\mathfrak{U}}^2 \times [0,1] \longrightarrow \mathcal{G}$ be an homotopy operator meeting the requirements listed below.

$(\tau_0)\ell \neq \mathfrak{L}_b(\ell,\varkappa,s)$, and$\varkappa \neq \mathfrak{L}_b(\ell,\varkappa,s)$for each $\ell,\varkappa \in \partial\mathfrak{U}$ and s∈ [0,1] (here $\partial\mathfrak{U}$ is boundary of $\mathfrak{U}$ in $\mathcal{G}$)

(τ_1) there exist $\ell, \varkappa, \text{æ}, \text{œ} \in \overline{\mathfrak{U}}$ and a$\in \mathfrak{A}$ with $\|\sqrt{2}a\| < 1$ such that $4kS_b(\mathfrak{L}_b(\ell,\varkappa,s),\mathfrak{L}_b(\ell,\varkappa,s),\mathfrak{L}_b(\text{æ},\text{œ},s)) \preccurlyeq a^*S_b(\ell,\ell,\text{æ})a + a^*S_b(\varkappa,\varkappa,\text{œ})a$ $(\tau_2)\exists\, M_b \succcurlyeq 0_{\mathfrak{U}}$such that $S_b(\mathfrak{L}_b(\ell,\varkappa,s),\mathfrak{L}_b(\ell,\varkappa,s),\mathfrak{L}_b(\ell,\varkappa,t)) \leq \|M_b\||s-t|$ for every $\ell,\varkappa \in \overline{\mathfrak{U}}$ and s, t $\in [0,1]$; $Then$ $\mathfrak{L}_b(,,s)$has a coupled fixed point for some s $\in [0,1] \Leftrightarrow \mathfrak{L}_b(,,t)$ has a coupled fixed point for some t $\in [0,1]$.

Proof: From (τ_2) it follows that $\mathfrak{L}_b$ is continuous in the second variable. From (τ_1) it follows that $\mathfrak{L}_b$ is continuous in the first variable. We have that

$$S_b\big(\mathfrak{L}_b(\ell,\varkappa,s),\mathfrak{L}_b(\ell,\varkappa,s),\mathfrak{L}_b(\ell_p,\varkappa_{\boldsymbol{p}},s)\big) \preccurlyeq 4kS_b\big(\mathfrak{L}_b(\ell,\varkappa,s),\mathfrak{L}_b(\ell,\varkappa,s),\mathfrak{L}_b(\ell_p,\varkappa_{\boldsymbol{p}},s)\big)$$
$$\preccurlyeq a^*S_b\big(\ell,\ell,\ell_p\big)a + a^*S_b\big(\varkappa,\varkappa,\varkappa_{\boldsymbol{p}}\big)a \longrightarrow 0_{\mathfrak{U}} \text{ as } p \longrightarrow \infty \;.$$

If $\ell_p \longrightarrow \ell$ and $\varkappa_{\boldsymbol{p}} \longrightarrow \varkappa$,then $S_b\big(\mathfrak{L}_b(\ell,\varkappa,s),\mathfrak{L}_b(\ell,\varkappa,s),\mathfrak{L}_b(\ell_p,\varkappa_{\boldsymbol{p}},s)\big) \longrightarrow 0_{\mathfrak{U}}$ as $p \longrightarrow \infty$.

Therefore,$S_b(\mathfrak{L}_b(\ell,\varkappa,s),\mathfrak{L}_b(\ell,\varkappa,s),\mathfrak{L}_b(\ell,\varkappa,s)) = 0_{\mathfrak{U}}$ and also we have $S_b(\mathfrak{L}_b(\varkappa,\ell,s),\mathfrak{L}_b(\varkappa,\ell,s),\mathfrak{L}_b(\varkappa,\ell,s)) = 0_{\mathfrak{U}}$

Now

$$S_b(\mathfrak{L}_b(\ell,\varkappa,s),\mathfrak{L}_b(\ell,\varkappa,s),\mathfrak{L}_b(\text{æ},\text{œ},t)) \preccurlyeq 2kS_b(\mathfrak{L}_b(\ell,\varkappa,s),\mathfrak{L}_b(\ell,\varkappa,s),\mathfrak{L}_b(\text{æ},\text{œ},s))$$
$$+kS_b(\mathfrak{L}_b(\text{æ},\text{œ},t),\mathfrak{L}_b(\text{æ},\text{œ},t),\mathfrak{L}_b(\text{æ},\text{œ},s))$$
$$\preccurlyeq 2kS_b\big(\mathfrak{L}_b(\ell,\varkappa,s),\mathfrak{L}_b(\ell,\varkappa,s),\mathfrak{L}_b(\text{æ},\text{œ},s)\big) + k\|M_b\||s-t|$$

$\preccurlyeq 2ka^*S_b(\ell,\ell,\text{æ})a + a^*S_b(\varkappa,\varkappa,\text{œ})a + k\|M_b\||s-t| \longrightarrow 0_{\mathfrak{U}}$ as $(\ell,\varkappa,s) \longrightarrow (\text{æ},\text{œ},t)$.

Hence $\mathfrak{L}_b$ is a continuous function on $\overline{\mathfrak{U}}^2 \times [0,1]$. Also

$$S_b(\mathfrak{L}(\ell,\varkappa,s),\mathfrak{L}_b(\ell,\varkappa,s),\mathfrak{L}_b(\text{æ},\text{œ},s)) \preccurlyeq 4kS_b(\mathfrak{L}_b(\ell,\varkappa,s),\mathfrak{L}_b(\ell,\varkappa,s),\mathfrak{L}_b(\text{æ},\text{œ},s))$$

$\preccurlyeq a^*S_b(\ell,\ell,\text{æ})a + a^*S_b(\varkappa,\varkappa,\text{œ})a$ if $\ell \neq \text{æ}$ **,**$\varkappa \neq \text{œ}$

$$\begin{pmatrix} S_b\big(\mathfrak{L}_b(\ell,\varkappa,s),\mathfrak{L}_b(\ell,\varkappa,s),\mathfrak{L}_b(\text{æ},\text{œ},s)\big) \\ +S_b(\mathfrak{L}_b(\varkappa,\ell,s),\mathfrak{L}_b(\varkappa,\ell,s),\mathfrak{L}_b(\text{œ},\text{æ},s)) \end{pmatrix} \prec \left(\sqrt{2}\,a\right)^*(S_b(\ell,\ell,\text{æ}) + S_b(\varkappa,\varkappa,\text{œ}))\left(\sqrt{2}\,a\right).$$

Now consider the set

$\mathfrak{B} = \{$s$\in [0,1] : \ell = \mathfrak{L}_b(\ell,\varkappa,s)$, $\varkappa = \mathfrak{L}_b(\varkappa,\ell,s)$ for some $\ell,\varkappa \in \mathfrak{U}\}$. Suppose s is a limit point of $\mathfrak{B}$ .Then there exists a $\{s_p\}$ in $\mathfrak{B}$ such that $s_p \longrightarrow s$ .Then there exists a sequence $\{\ell_p\},\{s_p\} \in \mathcal{G}$ such that $\ell_p = \mathfrak{L}_b\big(\ell_p,\varkappa_p,s_p\big)$ and $\varkappa_p = \mathfrak{L}_b\big(\varkappa_p,\ell_p,s_p\big)$ .Now we show that $\{\ell_p\},\{s_p\}$ are S_b-Cauchy sequences with respect $\mathfrak{A}$. So there exists $\epsilon \succ 0_{\mathfrak{A}}$ and monotonically increasing sequences of natural numbers $\{q_z\}$ and $\{p_z\}$ such that $p_z > q_z$.

$S_b\big(\ell_{q_z},\ell_{q_z},\ell_{p_z}\big) \succcurlyeq \epsilon$ $S_b\big(\varkappa_{q_z},\varkappa_{q_z},\varkappa_{p_z}\big) \succcurlyeq \epsilon$ ……… (2.1.7) and

$S_b\big(\ell_{q_z},\ell_{q_z},\ell_{p_{z-1}}\big) \prec \epsilon$,$S_b\big(\varkappa_{q_z},\varkappa_{q_z},\varkappa_{p_{z-1}}\big) \prec \epsilon$ …… (2.1.8)

From (2.1.7) and (2.1.8), we have $\epsilon \preccurlyeq S_b\big(\ell_{q_z},\ell_{q_z},\ell_{p_z}\big)$

$$\preccurlyeq 2\text{k}S_b\big(\ell_{q_z},\ell_{q_z},\ell_{q_{z+1}}\big) + \text{k}S_b\big(\ell_{q_{z+1}},\ell_{q_{z+1}},\ell_{p_z}\big)$$

Letting z$\longrightarrow \infty$ we have that $\frac{\epsilon}{\text{k}} \preccurlyeq \lim_{\text{z}\longrightarrow\infty} S_b\big(\ell_{q_{z+1}},\ell_{q_{z+1}},\ell_{p_z}\big)$ ……. (2.1.9)

$Suppose|s-s_0| < \epsilon$ and $\ell \in \overline{S_b(\ell_0,\delta)}$, $\ell \neq \ell_0$, $\varkappa \in \overline{S_b(\varkappa_0,\delta)}$,$\varkappa \neq \varkappa_0$ then

$$4kS_b(\mathfrak{L}_b(\ell,\varkappa,s_0),\mathfrak{L}_b(\ell,\varkappa,s_0),\mathfrak{L}_b(\ell_0,\varkappa_0,s_0)) \preccurlyeq a^*S_b(\ell,\ell,\ell_0)a + a^*S_b(\varkappa,\varkappa,\varkappa_0)a$$

$Therefore$,

$$\begin{pmatrix} \|S_b(\mathfrak{L}_b(\ell,\varkappa,s_0),\mathfrak{L}_b(\ell,\varkappa,s_0),\mathfrak{L}_b(\ell_0,\varkappa_0,s_0))\| \\ +\big\|S_b\big(\mathfrak{L}_b(\varkappa,\ell,s_0),\mathfrak{L}_b(\varkappa,\ell,s_0),\mathfrak{L}_b(\varkappa_0,\ell_0,s_0)\big)\big\| \end{pmatrix} < \frac{\|\sqrt{2}a\|^2}{\|4k\|}\begin{pmatrix} \|S_b(\ell,\ell,\ell_0)\| \\ +\|S_b(\varkappa,\varkappa,\varkappa_0)\| \end{pmatrix}$$

$< \frac{\delta}{2\|k\|}$ But $S_b(\mathfrak{L}_b(\ell,\varkappa,s),\mathfrak{L}_b(\ell,\varkappa,s),v_b(\ell_0,\varkappa_0,s_0)) \preccurlyeq$
$2kS_b(\mathfrak{L}_b(\ell,\varkappa,s),\mathfrak{L}_b(\ell,\varkappa,s),\mathfrak{L}_b(\ell,\varkappa,s_0)) + kS_b(\mathfrak{L}(\ell_0,\varkappa_0,s_0),\mathfrak{L}_b(\ell_0,\varkappa_0,s_0),\mathfrak{L}_b(\ell,\varkappa,s_0)) \preccurlyeq 2k\|M_b\||s-s_0|$
$+kS_b(\mathfrak{L}_b(\ell_0,\varkappa_0,s_0),\mathfrak{L}_b(\ell_0,\varkappa_0,s_0),\mathfrak{L}_b(\ell,\varkappa,s_0))$

Therefore

$\|S_b(\mathfrak{L}_b(\ell,\varkappa,s),\mathfrak{L}_b(\ell,\varkappa,s),\mathfrak{L}_b(\ell_0,\varkappa_0,s_0))\| \leq 2\|k\|\|M_b\||s-s_0| + \|k\|\frac{\delta}{\|2k\|} < 2\|k\|\|M_b\|\,\epsilon + \frac{\delta}{2} < 2\|k\|\|M_b\|\frac{\delta}{4\|k\|\|M_b\|} + \frac{\delta}{2} = \delta$

Hence

$$\|S_b(\mathfrak{L}_b(\ell,\varkappa,s),\mathfrak{L}_b(\ell,\varkappa,s),\ell_0)\| \leq \|S_b(\mathfrak{L}_b(\ell,\varkappa,s),\mathfrak{L}_b(\ell,\varkappa,s),\mathfrak{L}_b(\ell_0,\varkappa_0,s_0))\| \leq \delta$$

Therefore, $\mathfrak{L}_b(\ell,\varkappa,s) \in \overline{S_b(\ell_0,\delta)}$ and similarly, we prove that $\mathfrak{L}_b(\varkappa,\ell,s) \in \overline{S_b(\varkappa_0,\delta)}$.

Thus, for any s, with $|s-s_0| < \epsilon$ and s$\in [0,1]$, it follows that $\Gamma : \overline{S_b(\ell_0,\delta)} \times \overline{S_b(\ell_0,\delta)} \longrightarrow \overline{S_b(\ell_0,\delta)}$ defined by $\Gamma(\ell,\varkappa) = \mathfrak{L}_b(\ell,\varkappa,s)$ satisfies all the hypothesis of the theorem (5.1.6). Hence Γ has a coupled fixed point. i.e. $\Gamma(\ell,\varkappa) = \ell$ for some $\ell \in \overline{S_b(\ell_0,\delta)} \subseteq \mathfrak{U}$
$\Gamma(\varkappa,\ell) = \varkappa$ for some $\varkappa \in \overline{S_b(\varkappa_0,\delta)} \subseteq \mathfrak{U}$ therefore, $\Gamma(\ell,\varkappa) = \mathfrak{L}_b(\ell,\varkappa,s) = \ell$ and $\Gamma(\varkappa,\ell) = \mathfrak{L}_b(\varkappa,\ell,s) = \varkappa$and hence s$\in \mathfrak{B}$ .Thus $|s-s_0| < \epsilon \Rightarrow$ s$\in \mathfrak{B}$. But $\lim_{z\longrightarrow\infty} 4\text{k}S_b\big(\ell_{q_{z+1}},\ell_{q_{z+1}},\ell_{p_z}\big) =$

$$\lim_{z\longrightarrow\infty} 4\text{k}S_b\left(\mathfrak{L}_b\big(\ell_{q_{z+1}},\varkappa_{q_{z+1}},s_{q_{z+1}}\big),\mathfrak{L}_b\big(\ell_{q_{z+1}},\varkappa_{q_{z+1}},s_{q_{z+1}}\big),\mathfrak{L}_b\big(\ell_{p_z},\varkappa_{p_z},s_{p_z}\big)\right)$$
$$\preccurlyeq \lim_{z\longrightarrow\infty}\left(a^*S_b\big(\ell_{q_{z+1}},\ell_{q_{z+1}},\ell_{p_z}\big)a + a^*S_b\big(\varkappa_{q_{z+1}},\varkappa_{q_{z+1}},\varkappa_{p_z}\big)a\right)$$

It follows that

$$\lim_{z\longrightarrow\infty}\left(4\|k\| - \|\sqrt{2}a\|^2\right)\left(\big\|S_b\big(\ell_{q_{z+1}},\ell_{q_{z+1}},\ell_{p_z}\big)\big\| + \big\|S_b\big(\varkappa_{q_{z+1}},\varkappa_{q_{z+1}},\varkappa_{p_z}\big)\big\|\right) \leq 0$$

Thus $\lim_{z\longrightarrow\infty} S_b\big(\ell_{q_{z+1}},\ell_{q_{z+1}},\ell_{p_z}\big) = 0_{\mathfrak{U}}$ and $\lim_{z\longrightarrow\infty} S_b\big(\varkappa_{q_{z+1}},\varkappa_{q_{z+1}},\varkappa_{p_z}\big) = 0_{\mathfrak{U}}$. Hence from (2.1.9),
We have that $\epsilon \preccurlyeq 0_{\mathfrak{U}}$ which is a contradiction. Hence $\{\ell_p\}$ and $\{s_p\}$ are C^*-AV-S_b-CS in

C^*-AV-S_b-MS $(\mathcal{G}, \mathfrak{U}, S_b)$ and by the completeness of $(\mathcal{G}, \mathfrak{U}, S_b)$, there exist æ, œ $\in \mathfrak{U}$ with $\lim_{p\to\infty} \ell_p$=æ and $\lim_{p\to\infty} \varkappa_p$=œ . Suppose $s_p \longrightarrow s$,then $(\ell_p, \varkappa_p, s_p) \longrightarrow$ (æ, œ, s). Since $\mathfrak{L}_b$ is continuous so that
$\mathfrak{L}_b\ (\ell_p, \varkappa_p, s_p) \longrightarrow \mathfrak{L}_b($ æ, œ, $s)$ and as well as $\mathfrak{L}_b\ (\varkappa_p, \ell_p, s_p) \longrightarrow \mathfrak{L}_b($œ, æ, $s)$. But
$\mathfrak{L}_b\ (\ell_p, \varkappa_p, s_p) \longrightarrow \ell_p \longrightarrow$ æ and $\mathfrak{L}_b\ (\varkappa_p, \ell_p, s_p) \longrightarrow \varkappa_p \longrightarrow$ œ . Therefore, we have
$\mathfrak{L}_b\ (\ell_n, \varkappa_n, s_n) =$ æ and $\mathfrak{L}_b\ (\varkappa_n, \ell_n, s_n) =$ œ .Hence $\mathfrak{B}$ is closed.
Now we show that $\mathfrak{B}$ is open. Let $\mathfrak{B}$ be s_0 .Then ℓ_0, $\varkappa_0$ in $\mathfrak{U}$ such that $\ell_0 = \mathfrak{L}_b\ (\ell_0, \varkappa_0, s_0)$ and $\varkappa_0 = \mathfrak{L}_b\ (\varkappa_0, \ell_0, s_0)$ .Because $\mathfrak{U}$ is open ,$\delta > 0$ must exist for $S_b(\ell, \ell, \ell_0) \preccurlyeq \delta$and $S_b(\varkappa, \varkappa, \varkappa_0) \preccurlyeq \delta$implies that $\ell, \varkappa \in \mathfrak{U}$ .Choose ϵ such that $0<\epsilon < \frac{\delta}{4\|k\|\|M_b\|}$,Then s_0 is an interior point of $\mathfrak{B}$ .Hence $\mathfrak{B}$ is open . Consequently $\mathfrak{B}$ is both closed and open. Therefore, either $\mathfrak{B}=\emptyset$ or $\mathfrak{B}$ = [0,1] .Now Suppose $\mathfrak{L}_b(.\,;s)$ has a coupled fixed point for some s ∈ [0,1] ,then $\mathfrak{B} \neq \emptyset\ so\ that\ \mathfrak{B} =$ [0,1]$.Therefore\ \mathfrak{L}_b(.\,;t)$ has a coupled fixed point for all t ∈ [0,1] . A similar procedure can be used to demonstrate the opposite.

3. CONCLUSION

This paper presents certain applications of -generalized contractive type - coupled fixed point theorems within the framework of complete -algebra valued -metric spaces, specifically focusing on homotopy theory and integral equations.

ACKNOWLEDGEMENT

Author's would like to thank Department of Mathematics, Mahatma Gandhi University Nalgonda, Telangana, India for providing all the facilities and support.

Author contribution Statement

Conceptualization: G.Upender Reddy; **Literature Review and Methodology design:** N.Narsimha ; **Software:** P.Naresh ; **Validation:** G.Upender Reddy and P.Naresh; **Formal analysis:** N.Narsimha and P.Naresh ; **Investigation:** G.Upender Reddy ; **Resources:** G.Upender Reddy and P.Naresh ; **Data Curation:** N.Narsimha ; **Writing original draft preparation:** G.Upender Reddy;**Writing review and editing:** G.Upender Reddy;**Visualization:** P.Naresh; **Supervision:** G.Upender Reddy; **Project Administration:** G.Upender Reddy, All authors have read and agreed to the published version of the manuscript

Conflict of interest

The authors declare no conflicts of interest.

Data availability statement

Data supporting these findings are available within the article, at https://doi.org/10.26562/irjcs.2026.v1302.02, or upon request.

Publisher's note

REFERENCES

1. S.S. Razavi, H.P.Masiha, Common fixed -point theorems in C^*-algebra valued b-metric spaces with applications to integral equations, Fixed Point Theory 2019, 20, 649662. https://doi.org/10.30697/rfpta-2018-034
2. S.Sedghi, Gholidahneh, T.Dosenovic, J.Esfahani and S.Radenkovic, Common fixed point of four maps in S_b-metric spaces, Journal of Linear and Topological Algebra, Vol.5(2), (2016), 93- 104.
3. N.Souayah, N.Mlaiki A fixed point theorems in S_b-metric spaces, Journal of mathematics and computer science, 16(2016), 131-139. https://doi.org/10.22436/jmcs.016.02.01
4. Souayah Nizar, A fixed point in partial S_b-metric spaces, An. St.univovidius constante, vol24(3),2016, 315-362, doi: 10.1515/auom-2016-0062. https://doi.org/10.1515/auom-2016-0062
5. K.P.R.Sastry, K.K.M.Sarma, P. Krishna Kumari, Fixed point theorems for $(\psi, \varphi, \lambda$)- contractions in S_b-metric spaces, International Journal of Mathematics Trends and Technology (IJMTT) Volume 56(1), 2018, PP: 28-38. https://doi.org/10.14445/22315373/IJMTT-V56P505
6. Y.Rohen, T.Dosenovic, S.Radenovic, A Note on the Paper A Fixed Point Theorems in S_b-Metric Spaces, Filo mat. 31 (2017), 3335-3346. https://doi.org/10.2298/FIL1711335R
7. Duduka Venkatesh, V. Naga Raju, somefixed-point outcomes in S_b-metric spaces using $(\phi, \psi$)-generalized weakly contractive maps in S_b-metric spaces, Global Journal of Pure and Applied Mathematics, Volume 18, Number 2(2022), pp. 753-770.
8. Rohen y, Dose Nović T, Randanovic S,A note on a paper A fixed -point theorems in S_b-metric spaces, Filomat 31(11), 3335-3346(2017). https://doi.org/10.2298/FIL1711335R
9. K.P.R.Rao, G.N.V.Kishore, Sk.Sadik, Unique common coupled fixed- point theorem for four maps in S_b-metric spaces, Journal of Linear and Topological Algebra, Vol. 06, No. 01, 2017, 29- 43.
10. Kishore GNV, Rao KPR, Panthi D, Srinuvsa Rao B, Satyanaraya S, some applications via fixed point results in partially ordered S_b-metric spaces, Fixed point theory and applications (2017), 2017;10, https://doi.org/10.1186/s13663-017-0603-2

11. S.S.Razavi, H.P.Masiha, C^*-Algebra-Valued S_b-Metric Spaces and Applications to Integral Equations, Faculty of Mathematics, K.N. Toosi University of Technology, Tehran, Iran, 2023, in progress.
12. S.S. Razavi, H.P.Masiha, Manuel De La Sen, Applications in Integral Equations through Common Results in C-Algebra-Valued S_b-Metric Spaces, Axioms 2023, 12, 413. https://doi.org/10.3390/axioms12050413
13. D.J. Guo, V.Lakshmikantham, Coupled fixed points of nonlinear operators with applications, Nonlinear Anal., 11(1987), 623632. 1. https://doi.org/10.1016/0362-546X(87)90077-0
14. M.Abbas, M.Ali Khan, S.Radenovic, Common coupled fixed -point theorems in cone metric spaces forω-compatible mappings, Appl. Math. Comput., 217 (2010), 195202. 1. https://doi.org/10.1155/2010/134897
15. J.G. Mehta, M.L. Joshi, on coupled fixed point theorem in partially ordered complete metric space, Int. J. Pure Appl. Sci.Technol., 1 (2010),8792.